

# Aspects of general higher-order gravities

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## Abstract

We study several aspects of higher-order gravities constructed from general contractions of the Riemann tensor and the metric in arbitrary dimensions. First, we use the fast-linearization procedure presented in [arXiv:1607.06463](https://arxiv.org/abs/1607.06463) to obtain the equations satisfied by the metric perturbation modes on a maximally symmetric background in the presence of matter and to classify  $\mathcal{L}(\text{Riemann})$  theories according to their spectrum. Then, we linearize all theories up to quartic order in curvature and use this result to construct quartic versions of *Einsteinian cubic gravity* (ECG). In addition, we show that the most general cubic gravity constructed in a dimension-independent way and which does not propagate the ghost-like spin-2 mode (but can propagate the scalar) is a linear combination of  $f(\text{Lovelock})$  invariants, plus the ECG term, plus a *New ghost-free gravity* term. Next, we construct the generalized Newton potential and the Post-Newtonian parameter  $\gamma$  for general  $\mathcal{L}(\text{Riemann})$  gravities in arbitrary dimensions, unveiling some interesting differences with respect to the four-dimensional case. We also study the emission and propagation of gravitational radiation from sources for these theories in four dimensions, providing a generalized formula for the power emitted. Finally, we review Wald's formalism for general  $\mathcal{L}(\text{Riemann})$  theories and construct new explicit expressions for the relevant quantities involved. Many examples illustrate our calculations.

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# 1 Introduction & summary of results

Higher-order gravities have attracted a considerable amount of attention throughout the last decades. The reasons for this interest are manifold. On the one hand, whatever the *right* ultraviolet completion of Einstein gravity might turn out to be, the effective action of the theory is expected to contain a series of higher-derivative terms involving different contractions of the Riemann tensor and its covariant derivatives. This is naturally what happens in String Theory, which generically predicts the appearance of infinitely many of these subleading terms<sup>1</sup> correcting the Einstein-Hilbert (EH) action *e.g.*, [1–3].

Higher-curvature extensions of Einstein gravity have been extensively considered in the context of cosmology. In that case, the goal is going beyond the standard  $\Lambda$ -CDM model, *e.g.*, providing explanations for late-time accelerated expansion, dark matter or inflation — see *e.g.*, [4–7] for some reviews on the subject.

In the context of holography [8–10], higher-order gravities have also played a prominent role. In particular, they have been used as tools to characterize numerous properties of strongly coupled conformal field theories (CFTs), *e.g.*, [11–19]. In some cases, they have even been essential in the discovery of new universal results valid for general CFTs — holographic or not — [20–24].

Apart from these more or less well-delimited areas, another approach entails the identification and study of concrete classes of higher-order gravities which possess particularly interesting properties. In some cases they mimic defining aspects of Einstein gravity [25–29]. In others, they improve problematic characteristics of the theory — *e.g.*, by being renormalizable [30, 31]. More generally, the systematic study of higher-order gravities provides a deeper understanding of Einstein gravity itself, since it helps unveil what features of the theory are generic, and which ones are specific.

In this paper we will explore several aspects of gravity theories whose Lagrangian density is an arbitrary function of the Riemann tensor and the metric, *i.e.*,

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[ \mathcal{L}(R_{\mu\nu\rho\sigma}, g^{\alpha\beta}) + L_{\text{matter}} \right], \quad (1.1)$$

where we have included an additional term  $L_{\text{matter}}$  to account for possible additional minimally-coupled matter fields. Throughout the text, we shall refer to the class of theories defined by (1.1) as  $\mathcal{L}(\text{Riemann})$  gravities. While (1.1) does not account for the most general higher-order gravity conceivable<sup>2</sup>, it does incorporate a broad class of

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<sup>1</sup>Which terms appear depends on the particular set-up considered.

<sup>2</sup>Indeed, note that we shall not consider terms involving covariant derivatives of the Riemann tensor here. In fact, even that case would not encapsulate the most general theory if one considers the affine connection  $\Gamma_{\mu\nu}^{\rho}$  to be a dynamical field independent from the metric — à la Palatini — since that set-up allows for even richer scenarios — see [32–36] and references therein. Of course, similar comments apply if we introduce extra fields besides the metric, as in the case of scalar-tensor gravities — see *e.g.*, [37–39].

theories exhibiting very different features. Many aspects of general  $\mathcal{L}(\text{Riemann})$  theories have been previously developed in several contexts, including: black-hole mechanics, linearized gravity, holography or cosmology — see *e.g.*, [40–55] and references therein. We aim to develop some more here. In particular, we will perform a general and systematic study of the linearized spectrum of these theories, which we will use to compute relevant physical quantities such as the generalized Newtonian potential or the power radiated by sources. In addition, our classification will allow us to characterize some interesting previously unidentified theories. Finally, we will also study the Wald formalism for general  $\mathcal{L}(\text{Riemann})$  providing new explicit formulas for some of the relevant quantities involved<sup>3</sup>.

## 1.1 Main results

The main results of the paper can be summarized as follows:

- In section 2 we start by reviewing the fast-linearization procedure on maximally symmetric backgrounds (msb) presented in [29] and valid for general theories of the form (1.1) in general dimensions. This reduces the problem to the evaluation of the corresponding Lagrangian density on a particular Riemann tensor — constructed from the metric and an auxiliary tensor — and the computation of two trivial derivatives. We use this result to identify the physical modes propagated by the metric and the corresponding dynamical equations satisfied by those modes in the presence of matter in (Anti)-de Sitter and flat space. Finally, we construct an effective quadratic action from which the general linearized equations can be derived.
- In section 3 we classify all theories of the form (1.1) according to the properties of their physical modes. The categories include: theories which do not propagate an extra massive graviton but do incorporate a dynamical scalar; theories in which the extra graviton is present but the scalar is not; theories with two massless gravitons and a massive scalar, including generalized critical gravities — for which the scalar is absent — and Einstein-like theories, *i.e.*, those that only propagate a massless graviton.
- In section 4 we use our method to linearize the equations of motion of all theories contained in (1.1) up to quartic order in curvature in arbitrary dimensions.

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<sup>3</sup>Our conventions throughout the paper are as follows. We use  $(-, +, \dots, +)$  signature for the metric and the usual conventions [56] for the Riemann and Einstein tensors. We set  $\hbar = c = 1$  but keep the gravitational constant  $\kappa \equiv 8\pi G$  explicit. Very often we consider  $\kappa^{\frac{1}{D-2}}$  and  $\kappa^{\frac{1}{2-D}}$  to be the natural length and mass scales, respectively.

- In section 5 we explain how to obtain the linearized equations of a theory defined as a function of arbitrary curvature invariants starting from the linearized equations of each invariant. In particular, we prove that theories constructed as general functions of scalars whose linear combinations do not produce massive gravitons are also free of those modes.
- In section 6 we extend the construction of *Einsteinian cubic gravity* (ECG) [29] to quartic order. The resulting theories only propagate a massless graviton on a msb in general dimensions and they are defined in a dimension-independent manner, *i.e.*, the relative couplings between the different invariants involved are the same in all dimensions.
- In section 7 we construct the most general dimension-independent cubic theory of the form (1.1) which is free of massive gravitons in general dimensions — without imposing conditions on the extra scalar mode. This theory, which we call *New ghost-free gravity*, includes all the terms appearing in the ECG action — see (6.1) below — plus all  $f$ (Lovelock) invariants up to cubic order, plus a previously unidentified term which reads  $\mathcal{Y} \equiv R_{\mu}^{\alpha}{}_{\nu}{}^{\beta} R_{\alpha}^{\rho}{}_{\beta}{}^{\sigma} R_{\rho}^{\mu}{}_{\sigma}{}^{\nu} - 3R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + 2R_{\mu}^{\nu} R_{\nu}^{\rho} R_{\rho}^{\mu}$ . Just like the ECG term,  $\mathcal{Y}$  is non-trivial in four dimensions. As opposed to it, this new term does contribute to the denominator of the scalar mode mass<sup>4</sup>  $m_s$ .
- In section 8 we use the results in sections 2 and 3 to compute the generalized Newton potential  $U_D(r)$  and the *Parametrized Post-Newtonian* (PPN) parameter  $\gamma(r)$  for a theory of the form (1.1) in general dimensions. We show that  $U_D(r)$  takes the form of a combination of generalized Yukawa potentials which, for general  $D$  we show to be given by  $U_{D,\text{Yukawa}}(r) \sim (m/r)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(r)$ , where  $K_{\ell}(x)$  are modified Bessel functions of the second kind. We unveil interesting differences with respect to the four-dimensional case.
- In section 9 we use the results in sections 2 and 3 to study the emission and propagation of gravitational radiation from sources in general four-dimensional  $\mathcal{L}$ (Riemann) theories. We obtain general formulas for the radiative components of the different modes as well as for the total power emitted by a source in terms of the quadrupole moment and the scalar radiation. We apply these results to a binary system in a circular orbit.
- In section 10 we give a detailed account of Wald's formalism and construct explicit expressions for the relevant quantities involved for general  $\mathcal{L}$ (Riemann) theories.

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<sup>4</sup>Recall that none of the terms in the ECG action contribute to the denominator of  $m_s$ , which explains why there is no extra scalar in ECG [29].

New results are obtained for the symplectic structure  $\omega$  and the surface charge  $\delta Q_\xi - \xi \cdot \Theta$ .

- Finally, our appendices contain many examples which illustrate the results in sections 2, 3, 5 and 10.

## 2 Linearized equations of $\mathcal{L}(\text{Riemann})$ theories

In this section we study the linearized equations of general  $\mathcal{L}(\text{Riemann})$  theories on maximally symmetric backgrounds (msb) in arbitrary dimensions. The full non-linear equations of this class of theories (1.1) read [44]

$$\mathcal{E}_{\mu\nu} \equiv P_{\mu}{}^{\sigma\rho\lambda} R_{\nu\sigma\rho\lambda} - \frac{1}{2} g_{\mu\nu} \mathcal{L} - 2 \nabla^\alpha \nabla^\beta P_{\mu\alpha\beta\nu} = \frac{1}{2} T_{\mu\nu}, \quad (2.1)$$

where we defined the object

$$P^{\mu\nu\sigma\rho} \equiv \left[ \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\sigma}} \right]_{g^{\gamma\delta}}, \quad \text{and} \quad T_{\mu\nu} \equiv - \frac{2}{\sqrt{|g|}} \frac{\delta(\sqrt{|g|} L_{\text{matter}})}{\delta g^{\mu\nu}} \quad (2.2)$$

is the usual matter stress-energy tensor.

Our goal in this section is to review the fast-linearization procedure presented in [29] and explain how it can be used to characterize the spectrum of these theories, which we will use in numerous applications throughout the paper. In the first subsection we linearize (2.1) up to the identification of four constants  $a, b, c$  and  $e$ . We argue that those constants can be easily obtained from the corresponding Lagrangian following some simple steps that we detail. Then, we show that the general linearized equations can in fact be written in terms of only three physical parameters which can be easily obtained from  $a, b, c$  and  $e$ . These are nothing but the effective gravitational constant  $\kappa_{\text{eff}}$ , and the masses of the two extra modes which appear in the linearized spectrum of generic  $\mathcal{L}(\text{Riemann})$  theories,  $m_g^2$  and  $m_s^2$ . As we show, both in (Anti-)de Sitter and Minkowski backgrounds, the usual massless graviton is generically accompanied by a massive ghost-like graviton of mass  $m_g$  and a scalar mode of mass  $m_s$ . In subsection 2.3 we obtain the matter-coupled wave equations satisfied by these modes. We close the section by constructing a quadratic effective action from which the linearized equations can be obtained from the variation of the metric perturbation.

### 2.1 Linearization procedure

Let us start giving a detailed account of the fast-linearization method for general  $\mathcal{L}(\text{Riemann})$  theories presented in [29].

## First-order variations on a general background metric

Consider a perturbed metric of the form

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \quad (2.3)$$

where  $h_{\mu\nu} \ll 1$  for all  $\mu, \nu = 0, \dots, D-1$  and where  $\bar{g}_{\mu\nu}$  is any metric. Our goal is to expand the field equations (2.1) to linear order in  $h_{\mu\nu}$  assuming that  $\bar{g}_{\mu\nu}$  is a solution of the full non-linear ones. For this purpose, it is useful to define the tensor

$$C_{\sigma\rho\lambda\eta}^{\mu\gamma\sigma\nu} \equiv g_{\sigma\alpha} g_{\rho\beta} g_{\lambda\chi} g_{\eta\xi} \frac{\partial P^{\mu\gamma\sigma\nu}}{\partial R_{\alpha\beta\chi\xi}}, \quad (2.4)$$

where  $P^{\mu\nu\rho\sigma}$  was defined in (2.2). Now, using the identity [44]

$$\left[ \frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} \right]_{R_{\rho\sigma\gamma\delta}} = 2 P_{\mu}^{\rho\sigma\gamma} R_{\nu\rho\sigma\gamma}, \quad (2.5)$$

it is possible to prove that the variations of  $\mathcal{L}$  and  $P^{\mu\alpha\beta\nu}$  read respectively<sup>5</sup>

$$\begin{aligned} \delta \mathcal{L} &= 2\delta g^{\mu\nu} \bar{P}_{\mu}^{\sigma\rho\lambda} \bar{R}_{\nu\sigma\rho\lambda} + \bar{P}^{\mu\sigma\rho\lambda} \delta R_{\mu\sigma\rho\lambda}, \\ \delta P^{\mu\alpha\beta\nu} &= 2\delta g^{\lambda[\mu} \bar{P}_{\lambda}^{\alpha]\beta\nu} + 2\delta g^{\rho\eta} \bar{C}_{\lambda\eta\sigma\tau}^{\mu\alpha\beta\nu} \bar{R}_{\rho}^{\lambda\sigma\tau} + \bar{C}_{\lambda\rho\sigma\tau}^{\mu\alpha\beta\nu} \bar{g}^{\lambda\eta} \bar{g}^{\rho\gamma} \bar{g}^{\sigma\kappa} \bar{g}^{\tau\nu} \delta R_{\eta\gamma\sigma\tau}, \end{aligned} \quad (2.7)$$

where the bars mean evaluation on the background metric  $\bar{g}_{\mu\nu}$ .

## Maximally symmetric background

Since we are interested in the linearized version of (2.1) on an arbitrary msb  $(\bar{\mathcal{M}}, \bar{g}_{\mu\nu})$ , we will from now on assume that  $\bar{g}_{\mu\nu}$  satisfies

$$\bar{R}_{\mu\nu\alpha\beta} = 2\Lambda \bar{g}_{\mu[\alpha} \bar{g}_{\beta]\nu}, \quad (2.9)$$

for some constant  $\Lambda$ . Obviously, the explicit expressions of  $\bar{P}^{\mu\alpha\beta\nu}$  and  $\bar{C}_{\sigma\rho\lambda\eta}^{\mu\alpha\beta\nu}$  will depend on the particular Lagrangian  $\mathcal{L}$  considered. Observe however that when these objects

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<sup>5</sup>Observe that throughout the paper we choose  $\{R_{\mu\nu\rho\sigma}, g^{\gamma\delta}\}$  to be the fundamental variables in  $\mathcal{L}$ . As explained in [44, 57], all expressions obtained using these variables are consistent with alternative elections such as  $\{R_{\nu\rho\sigma}^{\mu}, g^{\alpha\beta}\}$  or  $\{R_{\mu\nu}^{\rho\sigma}\}$ . In particular, using the identities analogous to (2.5) obtained in [44] for the different elections of variables it is possible to show that (2.7) and (2.8) are correct independently of such election. For example, if we choose  $\{R_{\mu\nu}^{\rho\sigma}\}$ , (2.7) and (2.8) can be written as

$$\delta \mathcal{L} = \bar{P}_{\mu\nu}^{\rho\lambda} \delta R_{\rho\lambda}^{\mu\nu}, \quad \delta P^{\mu\alpha\beta\nu} = 2\delta g^{\lambda[\mu} \bar{P}_{\lambda}^{\alpha]\beta\nu} + \bar{C}_{\lambda\rho\sigma\tau}^{\mu\alpha\beta\nu} \bar{g}^{\lambda\eta} \bar{g}^{\rho\gamma} \delta R_{\eta\gamma}^{\kappa\nu}. \quad (2.6)$$

are evaluated on a msb, the resulting expressions can only contain terms involving combinations of  $\bar{g}_{\mu\nu}$ ,  $\bar{g}^{\mu\nu}$  and  $\delta^\nu_\mu$ . In addition, as it is clear from (2.2) and (2.4),  $P^{\mu\nu\rho\sigma}$  and  $C^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}$  inherit the symmetries of the Riemann tensors appearing in their definitions. This forces  $\bar{P}^{\mu\alpha\beta\nu}$  to be given by

$$\bar{P}^{\mu\alpha\beta\nu} = 2e\bar{g}^{\mu[\beta}\bar{g}^{\nu]\alpha}, \quad (2.10)$$

where the value of the constant  $e$  depends on the theory. Similarly,  $\bar{C}^{\mu\alpha\beta\nu}_{\sigma\rho\lambda\eta}$  is fully determined by three tensorial structures, namely

$$\begin{aligned} \bar{C}^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} = & a \left[ \delta^{[\sigma}_\mu \delta^{\rho]}_\alpha \delta^{[\lambda}_\beta \delta^{\eta]}_\nu + \delta^{[\lambda}_\mu \delta^{\eta]}_\alpha \delta^{[\sigma]}_\beta \delta^{\rho]}_\nu \right] + b [\bar{g}_{\mu\beta}\bar{g}_{\alpha\nu} - \bar{g}_{\mu\nu}\bar{g}_{\alpha\beta}] [\bar{g}^{\sigma\lambda}\bar{g}^{\rho\eta} - \bar{g}^{\sigma\eta}\bar{g}^{\rho\lambda}] \\ & + 4c \delta^{[\sigma}_{(\tau} \bar{g}^{\rho][\lambda} \delta^{\eta]}_\epsilon \delta^\tau_{[\mu} \bar{g}_{\alpha][\beta} \delta^\epsilon_{\nu]}, \end{aligned} \quad (2.11)$$

where the only theory-dependent quantities are in turn the constants  $a$ ,  $b$  and  $c$ .

### Background embedding equation

Imposing  $\bar{g}_{\mu\nu}$  to solve the field equations (2.1) with  $T_{\mu\nu} = 0$ , one finds

$$\bar{\mathcal{L}}(\Lambda) = 4e(D-1)\Lambda. \quad (2.12)$$

This is a relation between the background scale  $\Lambda$  defined in (2.9) and all the possible couplings appearing in the higher-order Lagrangian  $\mathcal{L}(\text{Riemann})$ . Another equation relating  $e$  and  $\Lambda$  can be obtained using (2.9) and (2.10). This reads in turn

$$\frac{d\bar{\mathcal{L}}(\Lambda)}{d\Lambda} = \bar{P}^{\mu\nu\rho\sigma} 2\bar{g}_{\mu[\rho}\bar{g}_{\sigma]\nu} = 2eD(D-1), \quad (2.13)$$

which, along with (2.12) produces the nice expression

$$\Lambda \frac{d\bar{\mathcal{L}}(\Lambda)}{d\Lambda} = \frac{D}{2} \bar{\mathcal{L}}(\Lambda). \quad (2.14)$$

This is the algebraic equation that needs to be solved in order to determine the possible vacua of the theory, *i.e.*, the allowed values of  $\Lambda$  as functions of the scales and couplings appearing in  $\mathcal{L}(\text{Riemann})$ <sup>6</sup>. Remarkably, (2.14) is also valid for theories involving general covariant derivatives of the Riemann tensor. Indeed, the most general higher-order gravity can be written as  $\mathcal{L}(R_{\mu\nu\rho\sigma}, \nabla_\alpha R_{\mu\nu\rho\sigma}, \nabla_\beta \nabla_\alpha R_{\mu\nu\rho\sigma}, \dots)$ . Now, maximally symmetric spaces have a covariantly constant Riemann tensor, so the derivatives of the Riemann do not have any effect on the background embedding equation. Therefore, (2.14) applies equally in such cases.

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<sup>6</sup>For example, for the Einstein-Hilbert action  $\mathcal{L} = R - 2\Lambda_0$ , (2.14) imposes  $\Lambda_0 = (D-1)(D-2)\Lambda/2$ . For Gauss-Bonnet with a negative cosmological constant  $\mathcal{L} = R + (D-1)(D-2)/L^2 + L^2\lambda_{\text{GB}}/((D-3)(D-4))\mathcal{X}_4$ , one finds the well-known relation  $-L^2\Lambda = (1 \pm \sqrt{1 - 4\lambda_{\text{GB}}})/(2\lambda_{\text{GB}})$ , see *e.g.*, [17].



## Linearization procedure

With the information from the previous items, we are ready to linearize (2.1). The result of a long computation in which we make use of (2.2)-(2.11) reads

$$\begin{aligned} \frac{1}{2}\mathcal{E}_{\mu\nu}^L = & + [e - 2\Lambda(a(D-1) + c) + (2a + c)\bar{\square}] G_{\mu\nu}^L + [a + 2b + c] [\bar{g}_{\mu\nu}\bar{\square} - \bar{\nabla}_\mu\bar{\nabla}_\nu] R^L \\ & - \Lambda[a(D-3) - 2b(D-1) - c] \bar{g}_{\mu\nu} R^L = \frac{1}{4}T_{\mu\nu}^L, \end{aligned} \quad (2.15)$$

where the linearized Einstein and Ricci tensors and the linearized Ricci scalar read, respectively<sup>7</sup>

$$G_{\mu\nu}^L = R_{\mu\nu}^L - \frac{1}{2}\bar{g}_{\mu\nu}R^L - (D-1)\Lambda h_{\mu\nu}, \quad (2.16)$$

$$R_{\mu\nu}^L = \bar{\nabla}_{(\mu}\bar{\nabla}_{\sigma}h^{\sigma}_{|\nu)} - \frac{1}{2}\bar{\square}h_{\mu\nu} - \frac{1}{2}\bar{\nabla}_\mu\bar{\nabla}_\nu h + D\Lambda h_{\mu\nu} - \Lambda h\bar{g}_{\mu\nu}, \quad (2.17)$$

$$R^L = \bar{\nabla}^\mu\bar{\nabla}^\nu h_{\mu\nu} - \bar{\square}h - (D-1)\Lambda h. \quad (2.18)$$

The above equations are quartic in derivatives of the perturbation for generic higher-derivative theories, as expected. The problem is hence reduced to the evaluation of  $a$ ,  $b$ ,  $c$  and  $e$  for a given theory, something that can be done using (2.2), (2.4), (2.10) and (2.11). However, this is a very tedious procedure in general, which involves the computation of first and second derivatives of  $\mathcal{L}(\text{Riemann})$  with respect to the Riemann tensor. The method presented in [29] allows for an important simplification of this problem. The procedure has several steps which we explain now.

1. Consider an auxiliary symmetric tensor  $k_{\mu\nu}$  satisfying

$$k^\mu{}_\mu = \chi, \quad k^\mu{}_\alpha k^\alpha{}_\nu = k^\mu{}_\nu, \quad (2.19)$$

where  $\chi$  is an arbitrary integer constant smaller than  $D$  which we will leave undetermined throughout the calculation. Note that the indices of  $k_{\mu\nu}$  are raised and lowered with  $g^{\mu\nu}$  and  $g_{\mu\nu}$ , as usual.

2. Define the following ‘‘Riemann tensor’’<sup>8</sup>

$$\tilde{R}_{\mu\nu\sigma\rho}(\Lambda, \alpha) \equiv 2\Lambda g_{\mu[\sigma}g_{\rho]\nu} + 2\alpha k_{\mu[\sigma}k_{\rho]\nu}, \quad (2.20)$$

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<sup>7</sup>Here we use the standard notation  $h \equiv \bar{g}^{\mu\nu}h_{\mu\nu}$ . Also, indices are raised and lowered with  $\bar{g}^{\mu\nu}$  and  $\bar{g}_{\mu\nu}$  respectively.

<sup>8</sup>The associated ‘‘Ricci tensor’’ and ‘‘Ricci scalar’’ are:  $\tilde{R}_{\mu\nu} = \Lambda(D-1)g_{\mu\nu} + \alpha(\chi-1)k_{\mu\nu}$  and  $\tilde{R} = \Lambda D(D-1) + \alpha\chi(\chi-1)$  respectively.

where  $\alpha$  and  $\Lambda$  are two parameters. Observe that  $\tilde{R}_{\mu\nu\sigma\rho}(\Lambda, \alpha)$  does not correspond — or more precisely, it does not need to correspond — to the Riemann tensor of any actual metric in general, even though it respects the symmetries of a true Riemann tensor. An exception occurs when  $\alpha = 0$ , as  $\tilde{R}_{\mu\nu\sigma\rho}(\Lambda, 0)$  becomes the Riemann tensor of a msb of curvature  $\Lambda$  associated to a metric  $g_{\mu\nu} = \bar{g}_{\mu\nu}$  as defined in (2.9).

3. Evaluate the higher-derivative Lagrangian (1.1) on  $\tilde{R}_{\mu\nu\sigma\rho}(\Lambda, \alpha)$ , *i.e.*, replace all Riemann tensors appearing in  $\mathcal{L}(\text{Riemann})$  by the object defined in (2.20). This gives rise to a function of  $\Lambda$  and  $\alpha$ <sup>9</sup>,

$$\mathcal{L}(\Lambda, \alpha) \equiv \mathcal{L}\left(R_{\mu\nu\rho\sigma} = \tilde{R}_{\mu\nu\rho\sigma}(\Lambda, \alpha), g^{\gamma\delta}\right). \quad (2.21)$$

4. The values of  $a, b, c$  and  $e$  can be obtained from the expressions

$$\left.\frac{\partial \mathcal{L}}{\partial \alpha}\right|_{\alpha=0} = 2e \chi(\chi - 1), \quad (2.22)$$

$$\left.\frac{\partial^2 \mathcal{L}}{\partial \alpha^2}\right|_{\alpha=0} = 4\chi(\chi - 1)(a + b\chi(\chi - 1) + c(\chi - 1)), \quad (2.23)$$

as can be proven using the chain rule along with equations (2.2), (2.4), (2.10) and (2.11). Interestingly, since  $a, b, c$  and  $e$  do not depend on  $\chi$  and they appear multiplied by factors involving different combinations of this parameter, we can identify them unambiguously for any theory by simple inspection. Once  $\mathcal{L}(\Lambda, \alpha)$  and its derivatives are computed, we just need to compare the resulting expressions with the RHS of (2.22) and (2.23) to obtain  $a, b, c$  and  $e$ <sup>1011</sup>.

5. Replace the values of  $a, b, c$  and  $e$  in the general expression (2.15).

This procedure is obviously simpler than computing  $\bar{P}^{\mu\nu\rho\sigma}$  and  $\bar{C}_{\lambda\eta\sigma\tau}^{\mu\nu\alpha\beta}$  explicitly using their definitions (2.2) and (2.4). Indeed, the most difficult step is the evaluation of

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<sup>9</sup>Note that in this evaluation, indices are still lowered with  $g_{\mu\nu}$ , and not with some combination of  $g_{\mu\nu}$  and  $k_{\mu\nu}$ .

<sup>10</sup>Observe that we only need  $\mathcal{L}(\Lambda, \alpha)$  up to  $\alpha^2$  order, *i.e.*, from  $\mathcal{L}(\Lambda, \alpha) = \mathcal{L}(\Lambda) + [2\chi(\chi - 1)e] \alpha + [2\chi(\chi - 1)(a + b\chi(\chi - 1) + c(\chi - 1))] \alpha^2 + \mathcal{O}(\alpha^3)$  we can read off the values of all the relevant constants.

<sup>11</sup>Equivalently, they can be obtained through direct evaluation of the following formulas,

$$e = \frac{1}{2\chi(\chi - 1)} \left.\frac{\partial \mathcal{L}}{\partial \alpha}\right|_{\alpha=0}, \quad a = \left[\frac{1}{4\chi(\chi - 1)} \left.\frac{\partial^2 \mathcal{L}}{\partial \alpha^2}\right|_{\alpha=0}\right] \Big|_{\chi=1}, \quad c = \left[\frac{1}{(\chi - 1)} \left[\frac{1}{4\chi(\chi - 1)} \left.\frac{\partial^2 \mathcal{L}}{\partial \alpha^2}\right|_{\alpha=0} - a\right]\right] \Big|_{\chi=0},$$

$$b = \frac{1}{\chi(\chi - 1)} \left[\frac{1}{4\chi(\chi - 1)} \left.\frac{\partial^2 \mathcal{L}}{\partial \alpha^2}\right|_{\alpha=0} - a - c(\chi - 1)\right],$$

where  $|_{\chi=1}$  means taking the limit  $\lim_{\chi \rightarrow 1}$  in the corresponding expression, etc.

$\mathcal{L}(\Lambda, \alpha)$ , which simply involves trivial contractions of  $g_{\mu\nu}$  and  $k_{\mu\nu}$  for any theory. The function  $\mathcal{L}(\Lambda, \alpha)$  is a sort of “prepotential” containing all the information needed for the linearization of a given higher-derivative theory of the form (1.1) on a msb.

We will apply this method in various sections of the paper — *e.g.*, see section 4 for the linearization of general quartic theories and section 5 for theories constructed as functions of curvature invariants. Appendix A contains a detailed application of our linearization procedure to quadratic theories and to a particular Born-Infeld-like theory.

Let us mention that in [50, 51, 58] a more refined method than the naive brute-force linearization of the full non-linear equations was also introduced for general  $\mathcal{L}(\text{Riemann})$  theories. This incorporates decompositions similar to the ones in (2.10) and (2.11), but still requires the somewhat tedious explicit evaluation of  $\bar{P}^{\mu\nu\rho\sigma}$  and  $\bar{C}^{\alpha\beta\gamma\delta}_{\mu\nu\rho\sigma}$  for each theory considered.

We close this subsection by mentioning that our linearization method reproduces all the particular cases previously studied in the literature. These include: quadratic gravities [27, 50, 58–60], Quasi-topological gravity [19, 61],  $f(R)$  [23] and general  $f(\text{Lovelock})$  theories [62].

## 2.2 Equivalent quadratic theory

The linearized equations (2.15) of any higher-order gravity of the form (1.1) characterized by some parameters  $a$ ,  $b$ ,  $c$  and  $e$ , can always be mapped to those of a quadratic theory of the form

$$\mathcal{L}_{\text{quadratic}} = \lambda(R - 2\Lambda_0) + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma \mathcal{X}_4, \quad (2.24)$$

where  $\mathcal{X}_4 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$  is the dimensionally-extended four-dimensional Euler density, also known as Gauss-Bonnet term. Indeed, the parameters  $\lambda$ ,  $\alpha$ ,  $\beta$  and  $\gamma$  of the equivalent quadratic theory can be obtained in terms of  $a$ ,  $b$ ,  $c$  and  $e$  through

$$\lambda = 2e - 4\Lambda[a + bD(D-1) + c(D-1)], \quad \alpha = 2b - a, \quad \beta = 4a + 2c, \quad \gamma = a. \quad (2.25)$$

Similarly, the cosmological constant  $\Lambda_0$  can be trivially related to the parameters appearing in (1.1) through  $\Lambda_0 = -\mathcal{L}(R_{\mu\nu\rho\sigma} = 0)/(2\lambda)$ .

Notice that the mapping from (1.1) to (2.24) is surjective but not injective, *i.e.*, all  $\mathcal{L}(\text{Riemann})$  theories are mapped to some quadratic theory, but (infinitely) many of them are mapped to the same one. Observe also that the existence of this mapping is a consequence of the fact that the linearized equations of any theory come from its action expanded at quadratic order in  $h_{\mu\nu}$  — see subsection 2.4. This means that the most general quadratic theory, namely (2.24) already contains all the possible kinds of terms produced in the action at order  $\mathcal{O}(h^2)$  of any  $\mathcal{L}(\text{Riemann})$  theory. Observe however that the fact that the parameters  $a$ ,  $b$ ,  $c$  and  $e$  for a given theory can be related to those appearing in (2.24) does not immediately help in identifying the values of those

parameters for a given theory. The mapping was explicitly performed for general cubic theories in [58].

## 2.3 Physical modes

As we just reviewed,  $\mathcal{E}_{\mu\nu}^L$  depends on four constants  $a$ ,  $b$ ,  $c$  and  $e$  as well as on the background curvature  $\Lambda$ . For a given theory, the four constants can be computed using the procedure explained in subsection 2.1, from which one can obtain the full linearized equations through (2.15). In this subsection we will explore how (2.15) can be further simplified using the gauge freedom of the metric perturbation and used to characterize the additional physical modes propagated by the metric in a general theory of the form (1.1).

Let us start with the following observation. If we parametrize  $a$ ,  $b$  and  $c$  in terms of three new constants  $m_g^2$ ,  $m_s^2$  and  $\kappa_{\text{eff}}$  as

$$\begin{aligned} a &= [4e\kappa_{\text{eff}} - 1] / [8\Lambda(D-3)\kappa_{\text{eff}}] , \\ b &= [(4e\kappa_{\text{eff}} - 1)(D-1)m_s^2m_g^2 + 2(3-2D+2(D-1)De\kappa_{\text{eff}})m_g^2\Lambda \\ &\quad + (D-3)\Lambda(Dm_s^2 + 4(D-1)\Lambda)] / [16\Lambda(D-3)\kappa_{\text{eff}}m_g^2(D-1)(m_s^2 + D\Lambda)] , \\ c &= -[(4e\kappa_{\text{eff}} - 1)m_g^2 + (D-3)\Lambda] / [4\Lambda(D-3)\kappa_{\text{eff}}m_g^2] , \end{aligned} \quad (2.26)$$

it is possible to rewrite (2.15) in terms of four different parameters, namely,  $\kappa_{\text{eff}}$ ,  $m_s^2$ ,  $m_g^2$  and  $\Lambda$ . Indeed, one finds

$$\begin{aligned} \mathcal{E}_{\mu\nu}^L &= \frac{1}{2\kappa_{\text{eff}}m_g^2} \left\{ [m_g^2 + 2\Lambda - \bar{\square}] G_{\mu\nu}^L + \left[ \frac{(D-2)(m_g^2 + m_s^2 + 2\Lambda)}{2(m_s^2 + D\Lambda)} \right] \Lambda \bar{g}_{\mu\nu} R^L \right. \\ &\quad \left. + \left[ \frac{(D-2)(m_g^2 - m_s^2 - 2(D-1)\Lambda)}{2(D-1)(m_s^2 + D\Lambda)} \right] [\bar{g}_{\mu\nu} \bar{\square} - \bar{\nabla}_\mu \bar{\nabla}_\nu] R^L \right\} = \frac{1}{2} T_{\mu\nu}^L , \end{aligned} \quad (2.27)$$

so the dependence on  $e$  disappears, while that on  $\kappa_{\text{eff}}$  gets factorized out from all terms. While (2.15) is more useful when computing the linearized equations of a particular theory — because we know a simple procedure to obtain  $a$ ,  $b$ ,  $c$  and  $e$  — (2.27) is more illuminating from a physical point of view. Indeed, as we will see in a moment,  $\kappa_{\text{eff}}$  will be the effective Einstein constant<sup>12</sup> while  $m_g^2$  and  $m_s^2$  will correspond, respectively, to the squared-masses of additional spin-2 and scalar modes.

It is straightforward to invert the relations (2.26) to obtain the values of such physical

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<sup>12</sup>Equivalently,  $\kappa_{\text{eff}} \equiv 8\pi G_{\text{eff}}$  where  $G_{\text{eff}}$  is the effective Newton constant.

quantities in terms of  $a$ ,  $b$ ,  $c$  and  $e$ . One finds

$$\kappa_{\text{eff}} = \frac{1}{4e - 8\Lambda(D-3)a}, \quad (2.28)$$

$$m_s^2 = \frac{e(D-2) - 4\Lambda(a + bD(D-1) + c(D-1))}{2a + Dc + 4b(D-1)}, \quad (2.29)$$

$$m_g^2 = \frac{-e + 2\Lambda(D-3)a}{2a + c}. \quad (2.30)$$

Let us stress that if we consider a theory consisting of a linear combination of invariants — like the one in (4.2) below — the values of  $a$ ,  $b$ ,  $c$  and  $e$  of that theory can be simply computed as the analogous linear combination of the parameters for each of those terms. However, that is not the case for  $\kappa_{\text{eff}}$ ,  $m_s^2$  and  $m_g^2$ , since they are not linear combinations of  $a$ ,  $b$ ,  $c$  and  $e$ . Hence, in order to determine these quantities for a given linear combination of invariants, the natural procedure should be obtaining the total values of  $a$ ,  $b$ ,  $c$  and  $e$  first, and then using (2.28)-(2.30) to compute the corresponding values of  $\kappa_{\text{eff}}$ ,  $m_s^2$  and  $m_g^2$ . For example, for a general quadratic theory of the form

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{(4-D)}{D-2}} (\alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}) \right], \quad (2.31)$$

the values of  $\kappa_{\text{eff}}$ ,  $m_g$  and  $m_s$  read, respectively

$$\kappa_{\text{eff}} = \frac{\kappa}{1 + 4\Lambda\kappa^{\frac{2}{D-2}} (\alpha_1 D(D-1) + \alpha_2(D-1) - 2\alpha_3(D-4))}, \quad (2.32)$$

$$m_s^2 = \frac{(D-2) + 4(D-4)\Lambda\kappa^{\frac{2}{D-2}} (\alpha_1 D(D-1) + \alpha_2(D-1) + 2\alpha_3)}{2\kappa^{\frac{2}{D-2}} (4\alpha_1(D-1) + \alpha_2 D + 4\alpha_3)}, \quad (2.33)$$

$$m_g^2 = \frac{-1 - 4\Lambda\kappa^{\frac{2}{D-2}} (\alpha_1 D(D-1) + \alpha_2(D-1) - 2\alpha_3(D-4))}{2\kappa^{\frac{2}{D-2}} (\alpha_2 + 4\alpha_3)}, \quad (2.34)$$

which we obtained using (2.28)-(2.30) and the values of  $a$ ,  $b$ ,  $c$  and  $e$  which appear in table 2. During the remainder of this section, we will write all expressions in terms of  $\kappa_{\text{eff}}$ ,  $m_s^2$  and  $m_g^2$ , which will make the presentation clearer. Nonetheless, all equations can be converted back to the language of  $a$ ,  $b$ ,  $c$  and  $e$  using the above relations.

The discussion proceeds slightly differently depending on whether we consider AdS/dS or Minkowski as the background spacetime, so we will consider the two cases separately. Let us start with the first.

### 2.3.1 (Anti-)de Sitter background

When studying the physical modes propagated by the metric perturbation on an AdS/dS background, it is customary and very convenient to work in the transverse gauge, in which<sup>13</sup>

$$\bar{\nabla}_\mu h^{\mu\nu} = \bar{\nabla}^\nu h. \quad (2.35)$$

Imposing this condition, many terms in (2.27) cancel out. Let us now expand the metric perturbation into its trace and traceless parts, which we denote by  $h$  and  $h_{\langle\mu\nu\rangle}$  respectively<sup>14</sup>,

$$h_{\mu\nu} = h_{\langle\mu\nu\rangle} + \frac{1}{D} \bar{g}_{\mu\nu} h. \quad (2.36)$$

Doing the same with the field equations (2.27), we find

$$\begin{aligned} \mathcal{E}_{\langle\mu\nu\rangle}^L = & + \frac{1}{2} T_{\langle\mu\nu\rangle}^L = \frac{1}{4m_g^2 \kappa_{\text{eff}}} \left\{ [\bar{\square} - 2\Lambda] [\bar{\square} - 2\Lambda - m_g^2] h_{\langle\mu\nu\rangle} - \bar{\nabla}_{\langle\nu} \bar{\nabla}_{\mu\rangle} \bar{\square} h \right. \\ & \left. + \left[ \frac{m_g^2(m_s^2 + 2(D-1)\Lambda) + \Lambda((4-3D)m_s^2 - 4(D-1)^2\Lambda)}{(m_s^2 + D\Lambda)} \right] \bar{\nabla}_{\langle\nu} \bar{\nabla}_{\mu\rangle} h \right\}, \end{aligned} \quad (2.37)$$

$$\mathcal{E}^L = + \frac{1}{2} T^L = - \left[ \frac{(D-1)(D-2)\Lambda(m_g^2 - (D-2)\Lambda)}{4\kappa_{\text{eff}} m_g^2(m_s^2 + D\Lambda)} \right] [\bar{\square} - m_s^2] h. \quad (2.38)$$

The second is the equation of motion of a free scalar field of mass  $m_s$ , while the first is an inhomogeneous equation for  $h_{\langle\mu\nu\rangle}$  as it involves also  $h$ . In order to obtain an independent equation for the traceless part, we define another traceless tensor:

$$t_{\mu\nu} \equiv h_{\langle\mu\nu\rangle} - \frac{\bar{\nabla}_{\langle\mu} \bar{\nabla}_{\nu\rangle} h}{(m_s^2 + D\Lambda)}, \quad (2.39)$$

where we have implicitly assumed that  $m_s^2 \neq -D\Lambda$ . After some manipulations, it can be seen that  $t_{\mu\nu}$  satisfies the equation

$$\frac{1}{2\kappa_{\text{eff}} m_g^2} (\bar{\square} - 2\Lambda)(\bar{\square} - 2\Lambda - m_g^2) t_{\mu\nu} = T_{\langle\mu\nu\rangle}^{L, \text{eff}}, \quad (2.40)$$

where we have defined the effective energy-momentum tensor

$$T_{\langle\mu\nu\rangle}^{L, \text{eff}} \equiv T_{\langle\mu\nu\rangle}^L + \frac{[\bar{\square} + (D-4)\Lambda - m_g^2] \bar{\nabla}_{\langle\mu} \bar{\nabla}_{\nu\rangle} T^L}{\Lambda(D-1)(D-2)(m_g^2 - (D-2)\Lambda)}. \quad (2.41)$$

<sup>13</sup>The metric decomposition performed in this section is similar to the one considered in [59].

<sup>14</sup>In this section, we denote the trace and traceless parts of rank-2 tensors  $P_{\mu\nu}$  linear in  $h_{\mu\nu}$  as  $P \equiv \bar{g}^{\mu\nu} P_{\mu\nu}$  and  $P_{\langle\mu\nu\rangle} \equiv P_{\mu\nu} - \frac{1}{D} \bar{g}_{\mu\nu} P$  respectively. In the case of the equations of motion, one can use the same notation, *i.e.*,  $\mathcal{E}^L \equiv \bar{g}^{\mu\nu} \mathcal{E}_{\mu\nu}^L$ ,  $T^L \equiv \bar{g}^{\mu\nu} T_{\mu\nu}^L$  — and similarly for the traceless part — because  $\bar{\mathcal{E}}_{\mu\nu} = \bar{T}_{\mu\nu} = 0$ . Observe however that  $R^L = (g^{\mu\nu} R_{\mu\nu})^L$  is not the trace of  $R_{\mu\nu}^L$ , but rather  $R^L = \bar{g}^{\mu\nu} R_{\mu\nu}^L - h^{\mu\nu} \bar{R}_{\mu\nu} = \bar{g}^{\mu\nu} R_{\mu\nu}^L - (D-1)h\Lambda$ .

Now, observe that the object

$$t_{\mu\nu}^{(m)} \equiv -\frac{1}{m_g^2}(\bar{\square} - 2\Lambda - m_g^2)t_{\mu\nu}, \quad (2.42)$$

satisfies the equation of the usual massless graviton, namely

$$-(\bar{\square} - 2\Lambda)t_{\mu\nu}^{(m)} = 2\kappa_{\text{eff}}T_{\langle\mu\nu\rangle}^{L,\text{eff}}, \quad (2.43)$$

but with a non-standard coupling to matter. On the other hand, using (2.42) and (2.43), it is easy to see that the tensor

$$t_{\mu\nu}^{(M)} \equiv t_{\mu\nu} - t_{\mu\nu}^{(m)} = \frac{1}{m_g^2}(\bar{\square} - 2\Lambda)t_{\mu\nu}, \quad (2.44)$$

satisfies instead

$$(\bar{\square} - 2\Lambda - m_g^2)t_{\mu\nu}^{(M)} = 2\kappa_{\text{eff}}T_{\langle\mu\nu\rangle}^{L,\text{eff}}. \quad (2.45)$$

Hence, we identify  $t_{\mu\nu}^{(M)}$  with a massive traceless spin-2 field with mass  $m_g$ . Observe that the coupling to matter of this mode has the wrong sign, which reflects its ghost-like behavior. Note that, apart from being a ghost, this mode is also tachyonic whenever  $m_g^2 < 0$ . The same occurs for the scalar when  $m_s^2 < 0$ .

In sum, using definitions (2.39), (2.42) and (2.44), we can decompose the metric perturbation  $h_{\mu\nu}$  as

$$h_{\mu\nu} = t_{\mu\nu}^{(m)} + t_{\mu\nu}^{(M)} + \frac{\bar{\nabla}_{\langle\mu}\bar{\nabla}_{\nu\rangle}h}{(m_s^2 + D\Lambda)} + \frac{1}{D}\bar{g}_{\mu\nu}h, \quad (2.46)$$

where  $h$ ,  $t_{\mu\nu}^{(M)}$  and  $t_{\mu\nu}^{(m)}$  satisfy (2.38), (2.45) and (2.43), and represent respectively: a scalar mode of mass  $m_s$ , a ghost-like spin-2 mode of mass  $m_g$  — which we will often refer to as “massive graviton” throughout the text — and a massless graviton.

### 2.3.2 Minkowski background

If we set  $\Lambda = 0$  in (2.38), this equation would lead us to conclude that  $T = 0$ . This inconsistency is a reflection of the fact that the transverse gauge can not be used in flat spacetime. The usual choice is in this case the so-called *de Donder gauge*, given by

$$\partial_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h. \quad (2.47)$$

In this gauge, the linearized field equations (2.27) in a Minkowski background can be written as

$$\mathcal{E}_{\mu\nu}^L = -\frac{1}{4\kappa_{\text{eff}}}\bar{\square}\hat{h}_{\mu\nu} = \frac{1}{2}T_{\mu\nu}^L, \quad (2.48)$$

where we have defined

$$\hat{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h - \frac{1}{m_g^2} \left[ \bar{\square} h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h \right] + \left[ \frac{m_g^2(D-2) + m_s^2}{2(D-1)m_g^2m_s^2} \right] [\eta_{\mu\nu}\bar{\square} - \partial_\mu\partial_\nu] h. \quad (2.49)$$

Using the gauge condition (2.47) it is easy to see that  $\hat{h}_{\mu\nu}$  is transverse, *i.e.*,

$$\partial_\mu \hat{h}^{\mu\nu} = 0. \quad (2.50)$$

Naturally,  $\hat{h}_{\mu\nu}$  is the usual spin-2 massless graviton, as it satisfies the linearized Einstein equation (2.48). However, there are more degrees of freedom (dof). In particular, we find that the metric can be decomposed as

$$\begin{aligned} h_{\mu\nu} = & \hat{h}_{\mu\nu} - \frac{1}{D-2}\eta_{\mu\nu}\hat{h} + \frac{1}{D-1}(m_g^{-2} - m_s^{-2})\partial_{\langle\mu}\partial_{\nu\rangle}\hat{h} \\ & + t_{\mu\nu} + \frac{2}{D(D-2)}\eta_{\mu\nu}\phi + \frac{1}{(D-1)m_s^2}\partial_{\langle\mu}\partial_{\nu\rangle}\phi, \end{aligned} \quad (2.51)$$

where  $t_{\mu\nu}$  is traceless and  $\phi$  is a scalar field. These objects satisfy the equations

$$-(\bar{\square} - m_s^2)\phi = 2\kappa_{\text{eff}}T^L, \quad (2.52)$$

$$(\bar{\square} - m_g^2)t_{\mu\nu} = 2\kappa_{\text{eff}} \left[ T_{\langle\mu\nu\rangle}^L + \frac{1}{(D-1)m_g^2}\partial_{\langle\mu}\partial_{\nu\rangle}T^L \right]. \quad (2.53)$$

Hence, even though we have proceeded in a different way as compared to the  $\Lambda \neq 0$  case, we have found the same physical modes: we have a massless spin-2 graviton  $\hat{h}_{\mu\nu}$ , a massive one  $t_{\mu\nu}$  and a scalar  $\phi$ , the masses of the last two being the same as the ones we found for  $t_{\mu\nu}^{(M)}$  and  $h$  in the (A)dS case. Note however that even though the dof and the masses are the same, the metric decomposition as well as the coupling of the fields to matter are different — compare (2.38) and (2.45) with (2.52) and (2.53), and (2.46) with (2.51). This can be understood as a consequence of the fact that the gauge which is convenient for (A)dS (2.35) differs from the de Donder one (2.47) utilized for Minkowski.

## 2.4 Quadratic action

As pointed out in section 2.2, the linearized equations (2.27) come from terms of order  $\mathcal{O}(h^2)$  in the action, which means that the structure of the linearized equations for the most general  $\mathcal{L}(\text{Riemann})$  is already captured by the most general quadratic theory. Expanding the action of a higher-order gravity to  $\mathcal{O}(h^2)$  is not trivial in general. However,



we can use the expression for the linearized equations (2.27) to find an action that yields these equations when varied with respect to  $h_{\mu\nu}$ . The easiest possibility is

$$S_2 = -\frac{1}{2} \int_{\mathcal{M}} d^D x h^{\mu\nu} \mathcal{E}_{\mu\nu}^L. \quad (2.54)$$

Using (2.27) and integrating by parts several times we find the effective action

$$S_2 = \int_{\mathcal{M}} \frac{d^D x}{4\kappa_{\text{eff}}} \left[ \frac{(D-2) [m_g^2 + (D-2)(m_s^2 + (D-1)\Lambda)]}{2(D-1)m_g^2(m_s^2 + D\Lambda)} (R^L)^2 - \left[ h^{\mu\nu} + \frac{2G^{L\mu\nu}}{m_g^2} \right] G_{\mu\nu}^L \right]. \quad (2.55)$$

As pointed out in [50], where an analogous action was found, (2.55) is manifestly invariant under “gauge” transformations  $h_{\mu\nu} \rightarrow h_{\mu\nu} + \bar{\nabla}_\mu \xi_\nu + \bar{\nabla}_\nu \xi_\mu$  as follows from the invariance of the linearized Einstein tensor and Ricci scalar under such transformations.

### 3 Classification of theories

In this section we will classify all gravity theories of the form (1.1) according to the properties of their physical modes. Indeed, depending on the values of the parameters  $a$ ,  $b$ ,  $c$  and  $e$ , we will divide them into five classes<sup>15</sup>: 1) theories without massive gravitons, *i.e.*, those for which the additional spin-2 mode is absent but the spin-0 one is dynamical; 2) theories without dynamical scalar, *i.e.*, those for which the additional graviton is dynamical but the spin-0 mode is absent; 3) theories with two massless gravitons and a massive scalar, *i.e.*, those for which the extra graviton is massless — a property which to some extent cures its problematic behavior; 4) generalized *critical* gravities *i.e.*, those which belong to the previous category and, in addition, have no additional spin-0 mode; 5) and finally, Einstein-like theories, *i.e.*, theories for which the only mode is the usual massless graviton. A summary of the different cases can be found in table 1. Various examples of particular theories belonging to each class are provided in appendix B. Let us also parenthetically mention that related analyses were previously performed in the absence of matter in [29, 50, 58].

	$m_g^2 = 0$	$0 < m_g^2 < +\infty$	$m_g^2 = +\infty$
$0 \leq m_s^2 < +\infty$	Massless gravitons + scalar	General case	No massive graviton
$m_s^2 = +\infty$	Critical	No dynamical scalar	Einstein-like

Table 1: Classification of theories according to their spectrum on a msb.

<sup>15</sup>Or six, if we count the general case in which  $m_g^2$  is finite and different from zero, and  $0 \leq m_s^2 < +\infty$ .

### 3.1 Theories without massive graviton

The ghost-like massive spin-2 mode  $t_{\mu\nu}^{(M)}$  found in the previous section can be removed from the linearized spectrum of the theory by imposing  $m_g^2 = +\infty$ . In terms of the parameters characterizing a given higher-derivative theory as described in section 2, such condition will be satisfied whenever

$$2a + c = 0. \quad (3.1)$$

When this condition holds, the linearized equations (2.27) become

$$\mathcal{E}_{\mu\nu}^L = \frac{1}{2\kappa_{\text{eff}}} \left\{ G_{\mu\nu}^L + \left[ \frac{(D-2)}{2(D-1)(m_s^2 + D\Lambda)} \right] [(D-1)\Lambda\bar{g}_{\mu\nu} + \bar{g}_{\mu\nu}\bar{\square} - \bar{\nabla}_\mu\bar{\nabla}_\nu] R^L \right\}. \quad (3.2)$$

Observe that (3.1) has the effect of making the  $\bar{\square}G_{\mu\nu}^L$  term — responsible for the appearance of the extra spin-2 graviton — disappear. As a consequence, even though (3.2) still contains quartic derivatives of  $h_{\mu\nu}$ , the equations do become second-order when we choose the transverse gauge  $\bar{\nabla}^\mu h_{\mu\nu} = \bar{\nabla}_\nu h$ , as it can be immediately checked from (3.2) using (2.18) — or alternatively from (2.37) taking the limit  $m_g^2 \rightarrow +\infty$  there.

On AdS/dS backgrounds — the extension to Minkowski is straightforward — (3.1) imposes  $t_{\mu\nu}^{(M)} = 0$ , so the metric decomposition becomes now

$$h_{\mu\nu} = t_{\mu\nu}^{(m)} + \frac{\bar{\nabla}_{(\mu}\bar{\nabla}_{\nu)}h}{(m_s^2 + D\Lambda)} + \frac{1}{D}\bar{g}_{\mu\nu}h, \quad (3.3)$$

where  $h$  and  $t_{\mu\nu}^{(m)}$  still satisfy (2.38) and (2.43) respectively. Observe that using (2.38) and (3.3) along with the transverse gauge condition (2.35), it is possible to show that  $t_{\mu\nu}^{(m)}$  is transverse in the vacuum,

$$\bar{\nabla}^\mu t_{\mu\nu}^{(m)} = 0. \quad (3.4)$$

Notice also that after imposing (2.35) we still have some gauge freedom, because a gauge transformation  $h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\bar{\nabla}_{(\mu}\xi_{\nu)}$  for any vector  $\xi_\mu$  satisfying  $\bar{\nabla}^\mu\bar{\nabla}_{(\mu}\xi_{\nu)} = \bar{\nabla}_\nu\bar{\nabla}_\mu\xi^\mu$  preserves (2.35). This allows us to impose additional conditions on  $h_{\mu\nu}$ . In particular, we can choose

$$t_{0\mu}^{(m)} = t_{\mu 0}^{(m)} = 0, \quad (3.5)$$

so that only the spatial components  $t_{ij}^{(m)}$ ,  $i, j = 1, \dots, D-1$  are non-zero. Then this tensor has  $D(D-1)/2$  components, but we have also

$$\bar{\nabla}^i t_{ij}^{(m)} = 0, \quad \bar{g}^{ij} t_{ij}^{(m)} = 0, \quad (3.6)$$

which follow from (3.4) and the tracelessness of  $t_{\mu\nu}^{(m)}$  respectively. These are  $(D-1)+1 = D$  constraints, so the number of polarizations of  $t_{\mu\nu}^{(m)}$  is  $D(D-3)/2$ , just like for the usual Einstein graviton. Of course, the trace  $h$  provides an additional degree of freedom, so these theories propagate  $(D-1)(D-2)/2$  physical dof in the vacuum.

### 3.2 Theories without dynamical scalar

The condition for the absence of the scalar mode is naturally given by  $m_s^2 = +\infty$ . In terms of the parameters  $a$ ,  $b$ ,  $c$  and  $e$ , this reads

$$2a + Dc + 4b(D - 1) = 0. \quad (3.7)$$

The linearized equations of motion (2.27) become in that case

$$\mathcal{E}_{\mu\nu}^L = \frac{1}{2\kappa_{\text{eff}}m_g^2} \left\{ [m_g^2 + 2\Lambda - \bar{\square}] G_{\mu\nu}^L + \frac{(D-2)}{2(D-1)} [(D-1)\Lambda\bar{g}_{\mu\nu} - \bar{g}_{\mu\nu}\bar{\square} + \bar{\nabla}_\mu\bar{\nabla}_\nu] R^L \right\}. \quad (3.8)$$

The metric decomposition simplifies to

$$h_{\mu\nu} = t_{\mu\nu}^{(m)} + t_{\mu\nu}^{(M)} + \frac{1}{D}\bar{g}_{\mu\nu}h, \quad (3.9)$$

where the trace of the metric perturbation is simply determined by the matter stress-tensor through the expression

$$h = \frac{2\kappa_{\text{eff}}m_g^2}{(D-1)(D-2)\Lambda(m_g^2 - (D-2)\Lambda)} T^L. \quad (3.10)$$

The massless and massive gravitons satisfy the same equations as in the general case, *i.e.*, (2.43) and (2.45) respectively.

### 3.3 Theories with two massless gravitons

As we saw,  $t_{\mu\nu}^{(M)}$  is a ghost. In order to remove this instability, the simplest solution is to consider theories in which it is absent. Another possibility is to set  $m_g = 0$ , namely, impose its mass to be zero like for the usual graviton. The condition to be satisfied is in this case

$$-e + 2\Lambda(D-3)a = 0. \quad (3.11)$$

From (2.28) we learn that (3.11) also imposes the effective Einstein constant to diverge,  $\kappa_{\text{eff}} = +\infty$ . This inconsistency is artificial and comes from a wrong identification of  $\kappa_{\text{eff}}$  in this case. In fact, the effective gravitational constant must be defined now as

$$\hat{\kappa}_{\text{eff}} \equiv m_g^2\kappa_{\text{eff}} = -\frac{1}{4(2a+c)}, \quad (3.12)$$

which remains finite when we impose (3.11). Then, the equation for the trace reads

$$\left[ \frac{(D-1)(D-2)^2\Lambda^2}{2\hat{\kappa}_{\text{eff}}(m_s^2 + D\Lambda)} \right] [\bar{\square} - m_s^2] h = T^L. \quad (3.13)$$

On the other hand, we cannot decompose the traceless perturbation  $t_{\mu\nu}$  into two independent fields. Instead, it fulfills the equation

$$\frac{1}{2\hat{\kappa}_{\text{eff}}}(\bar{\square} - 2\Lambda)^2 t_{\mu\nu} = T_{\langle\mu\nu\rangle}^{L,\text{eff}}, \quad (3.14)$$

with a metric decomposition given now by

$$h_{\mu\nu} = t_{\mu\nu} + \frac{\bar{\nabla}_{\langle\mu}\bar{\nabla}_{\nu\rangle}h}{(m_s^2 + D\Lambda)} + \frac{1}{D}\bar{g}_{\mu\nu}h. \quad (3.15)$$

### 3.4 Critical gravities

Critical gravities [27] are theories in which the extra graviton is massless and, in addition, the scalar mode is absent, *i.e.*, it satisfies  $m_s^2 = +\infty$ . Then, as shown in [27] for quadratic gravities in  $D = 4$ , the energies of both  $t_{\mu\nu}^{(m)}$  and  $t_{\mu\nu}^{(M)}$  become zero. The conditions to be imposed now are (3.11) and (3.7) as well as the redefinition of the Einstein constant in (3.12).

Then, the traceless part of the metric satisfies

$$\frac{1}{2\hat{\kappa}_{\text{eff}}}[(\bar{\square} - 2\Lambda)^2 h_{\langle\mu\nu\rangle} - \bar{\nabla}_{\langle\nu}\bar{\nabla}_{\mu\rangle}\bar{\square}h] = T_{\langle\mu\nu\rangle}^L, \quad (3.16)$$

while the trace is determined by matter,

$$h = -\frac{2\hat{\kappa}_{\text{eff}}}{(D-1)(D-2)^2\Lambda^2}T^L. \quad (3.17)$$

### 3.5 Einstein-like theories

When both the massive graviton and the scalar mode are absent, we are left with a theory whose only propagating degree of freedom is a massless graviton. The conditions  $m_g^2 = m_s^2 = +\infty$  can be expressed as

$$2a + c = 4b + c = 0. \quad (3.18)$$

The linearized equations of motion drastically simplify and become identical to those of Einstein gravity with an effective Einstein constant,

$$\mathcal{E}_{\mu\nu}^L = \frac{1}{2\kappa_{\text{eff}}}G_{\mu\nu}^L = \frac{1}{2}T_{\mu\nu}^L. \quad (3.19)$$

The metric decomposition is very simple now,

$$h_{\mu\nu} = t_{\mu\nu}^{(m)} + \frac{1}{D}\bar{g}_{\mu\nu}h, \quad (3.20)$$

with  $t_{\mu\nu}^{(m)}$  satisfying (2.43), and  $h$  being again completely determined by matter,

$$h = \frac{2\kappa_{\text{eff}}}{\Lambda(D-1)(D-2)} T^L. \quad (3.21)$$

Hence, according to the discussion in 3.1, the only propagating mode is the transverse and traceless part of the metric perturbation, which carries  $D(D-3)/2$  dof, like in Einstein gravity<sup>16</sup>.

## 4 Linearization of all theories up to quartic order

Up to quartic order in curvature, the most general  $D$ -dimensional theory of the form (1.1) can be written as

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} \sum_{i=1}^3 \alpha_i \mathcal{L}_i^{(2)} + \kappa^{\frac{6-D}{D-2}} \sum_{i=1}^8 \beta_i \mathcal{L}_i^{(3)} \right. \\ \left. + \kappa^{\frac{8-D}{D-2}} \sum_{i=1}^{26} \gamma_i \mathcal{L}_i^{(4)} \right\}. \quad (4.1)$$

Here,  $\mathcal{L}_i^{(2)}$ ,  $\mathcal{L}_i^{(3)}$  and  $\mathcal{L}_i^{(4)}$  represent, respectively, the quadratic, cubic and quartic curvature invariants enumerated in table 2,  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  are dimensionless constants and  $\kappa = 8\pi G$  is again Einstein's constant. Also  $\Lambda_0$  is the cosmological constant and we choose  $\kappa^{\frac{1}{D-2}}$  to be the natural scale<sup>17</sup>. In general dimensions there are three independent quadratic invariants, eight cubic and twenty-six quartic [63]. Naturally, these numbers get reduced as we consider small enough  $D$ . For example, in  $D = 4$  there are only two quadratic, six cubic and thirteen quartic invariants.

Using the procedure explained in section 2 we have linearized the quartic action (4.2), *i.e.*, we have computed the quantity  $\mathcal{L}(\Lambda, \alpha)$  defined in (2.21) at order  $\mathcal{O}(\alpha^2)$  for every term in the action and obtained the values of  $a$ ,  $b$ ,  $c$  and  $e$  from there. The results are shown in table 2. Finally, the parameters  $a$ ,  $b$ ,  $c$  and  $e$  of the full theory (4.2) can be found by adding linearly the contribution of each term, with the corresponding coefficients in front in each case, namely

$$e = \frac{1}{2\kappa} e[R] + \kappa^{\frac{4-D}{D-2}} \sum_{i=1}^3 \alpha_i e[\mathcal{L}_i^{(2)}] + \kappa^{\frac{6-D}{D-2}} \sum_{i=1}^8 \beta_i e[\mathcal{L}_i^{(3)}] + \kappa^{\frac{8-D}{D-2}} \sum_{i=1}^{26} \gamma_i e[\mathcal{L}_i^{(4)}], \quad (4.2)$$

<sup>16</sup>Note that throughout the text we use the labels *Einstein-like* and *Einsteinian* with slightly different meanings. By Einstein-like theories we mean theories for which the extra modes are absent and the only dynamical field at the linearized level is the usual massless graviton of general relativity. By Einsteinian we refer to those Einstein-like theories which are defined in a dimension-independent way — see section 6.

<sup>17</sup>This election can be trivially changed by a rescaling of the couplings, *e.g.*,  $\alpha_i \rightarrow \alpha_i / (\Lambda_0 \kappa^{\frac{2}{D-2}})^{\frac{4-D}{2}}$ .

where *e.g.*,  $e[R] = 1/2$  is the value of  $e$  corresponding to the Einstein-Hilbert term  $R$ , and so on. Completely analogous expressions hold for  $a$ ,  $b$  and  $c$ .

Table 2 along with the results in section 3 can be easily used to classify the different theories in (4.2) according to their spectrum.

## 5 $f(\text{scalars})$ theories

In section 4 we linearized all higher-derivative gravities of the form (1.1) up to quartic order. That class includes linear combinations of scalars  $\mathcal{R}_i$  constructed from contractions of the Riemann tensor and the metric, but not theories constructed as arbitrary functions of those scalars, such as  $f(R)$  gravity. In this section we will consider the latter case, *i.e.*, we will linearize the equations of motion of a theory of the form

$$\mathcal{L} = f(\mathcal{R}_1, \dots, \mathcal{R}_m), \quad (5.1)$$

where the  $\mathcal{R}_i$  are arbitrary scalars.

For a theory of this form, using the objects

$$P_i^{\mu\alpha\beta\nu} \equiv \frac{\partial \mathcal{R}_i}{\partial R_{\mu\alpha\beta\nu}}, \quad C_i^{\mu\gamma\sigma\nu}{}_{\sigma\rho\lambda\eta} \equiv g_{\sigma\alpha} g_{\rho\beta} g_{\lambda\chi} g_{\eta\xi} \frac{\partial P_i^{\mu\gamma\sigma\nu}}{\partial R_{\alpha\beta\chi\xi}}, \quad (5.2)$$

we get the following result for the tensors defined in (2.2) and (2.4) evaluated on the background,

$$\bar{P}^{\mu\alpha\beta\nu} = \partial_i f(\bar{\mathcal{R}}) \bar{P}_i^{\mu\alpha\beta\nu}, \quad \bar{C}^{\mu\alpha\beta\nu}{}_{\sigma\rho\lambda\eta} = \partial_i f(\bar{\mathcal{R}}) \bar{C}_i^{\mu\alpha\beta\nu}{}_{\sigma\rho\lambda\eta} + \partial_i \partial_j f(\bar{\mathcal{R}}) \bar{P}_i^{\mu\alpha\beta\nu} \bar{P}_j{}_{\sigma\rho\lambda\eta}, \quad (5.3)$$

where  $\partial_i$  denotes derivative with respect to  $\mathcal{R}_i$ , and  $\bar{\mathcal{R}}$  means that we evaluate all the scalars on the background. Using these expressions it is possible to obtain the values of the parameters  $a, b, c$  and  $e$  defined in (2.11) and (2.10) for the theory (5.1). The result is

$$a = \partial_i f(\bar{\mathcal{R}}) a_i, \quad b = \partial_i f(\bar{\mathcal{R}}) b_i + \partial_i \partial_j f(\bar{\mathcal{R}}) e_i e_j, \quad c = \partial_i f(\bar{\mathcal{R}}) c_i \quad e = \partial_i f(\bar{\mathcal{R}}) e_i. \quad (5.4)$$

Hence, once we have computed the parameters  $a_i, b_i, c_i, e_i$  for the set of scalars  $\mathcal{R}_i$ , we can easily find the corresponding parameters for any other Lagrangian  $\mathcal{L} = f(\mathcal{R}_1, \dots, \mathcal{R}_m)$ . Plugging the values (5.4) in (2.15), we obtain the linearized equations.

### 5.1 Theories without massive graviton

In section 3 we classified general  $\mathcal{L}(\text{Riemann})$  theories according to their spectrum on a msb. One of the cases under consideration was that corresponding to theories for

Label	Term	$e$	$a$	$b$	$c$
$\mathcal{L}_1^{(1)}$	$R$	$\frac{1}{2}$	0	0	0
$\mathcal{L}_1^{(2)}$	$R^2$	$D(D-1)\Lambda$	0	$\frac{1}{2}$	0
$\mathcal{L}_2^{(2)}$	$R_{\mu\nu}R^{\mu\nu}$	$(D-1)\Lambda$	0	0	$\frac{1}{2}$
$\mathcal{L}_3^{(2)}$	$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$	$2\Lambda$	1	0	0
$\mathcal{L}_1^{(3)}$	$R_{\mu}^{\rho}R_{\nu}^{\sigma}R_{\rho}^{\delta}R_{\sigma}^{\gamma}R_{\delta}^{\mu}R_{\gamma}^{\nu}$	$\frac{3}{2}(D-2)\Lambda^2$	$-\frac{3}{2}\Lambda$	0	$\frac{3}{2}\Lambda$
$\mathcal{L}_2^{(3)}$	$R_{\mu\nu}^{\rho\sigma}R_{\rho\sigma}^{\delta\gamma}R_{\delta\gamma}^{\mu\nu}$	$6\Lambda^2$	$6\Lambda$	0	0
$\mathcal{L}_3^{(3)}$	$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho}{}_{\delta}R^{\sigma\delta}$	$3(D-1)\Lambda^2$	$(D-1)\Lambda$	0	$2\Lambda$
$\mathcal{L}_4^{(3)}$	$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}R$	$3D(D-1)\Lambda^2$	$D(D-1)\Lambda$	$2\Lambda$	0
$\mathcal{L}_5^{(3)}$	$R_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}$	$\frac{3}{2}(D-1)^2\Lambda^2$	0	$\frac{1}{2}\Lambda$	$\frac{1}{2}(2D-3)\Lambda$
$\mathcal{L}_6^{(3)}$	$R_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\mu}$	$\frac{3}{2}(D-1)^2\Lambda^2$	0	0	$\frac{3}{2}(D-1)\Lambda$
$\mathcal{L}_7^{(3)}$	$R_{\mu\nu}R^{\mu\nu}R$	$\frac{3}{2}D(D-1)^2\Lambda^2$	0	$(D-1)\Lambda$	$\frac{1}{2}D(D-1)\Lambda$
$\mathcal{L}_8^{(3)}$	$R^3$	$\frac{3}{2}D^2(D-1)^2\Lambda^2$	0	$\frac{3}{2}D(D-1)\Lambda$	0
$\mathcal{L}_1^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu}^{\delta}R_{\nu}^{\gamma}R_{\rho}^{\chi}R_{\sigma}^{\xi}R_{\gamma\chi\sigma\xi}$	$2(3D-5)\Lambda^3$	$2(D-4)\Lambda^2$	0	$7\Lambda^2$
$\mathcal{L}_2^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu}^{\delta}R_{\nu}^{\gamma}R_{\rho}^{\chi}R_{\sigma}^{\xi}R_{\nu\chi\sigma\xi}$	$2(D^2-3D+4)\Lambda^3$	$6\Lambda^2$	$\Lambda^2$	$2(D-3)\Lambda^2$
$\mathcal{L}_3^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\delta\gamma}R_{\rho}^{\chi}R_{\sigma}^{\xi}R_{\sigma\chi\gamma\xi}$	$4(D-2)\Lambda^3$	$(D-7)\Lambda^2$	0	$5\Lambda^2$
$\mathcal{L}_4^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\delta\gamma}R_{\rho\delta}^{\chi\xi}R_{\sigma\gamma\chi\xi}$	$8\Lambda^3$	$12\Lambda^2$	0	0
$\mathcal{L}_5^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\delta\gamma}R_{\delta\gamma}^{\chi\xi}R_{\rho\sigma\chi\xi}$	$16\Lambda^3$	$24\Lambda^2$	0	0
$\mathcal{L}_6^{(4)}$	$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho}^{\delta}R_{\sigma}^{\gamma}R_{\delta\gamma}^{\chi\xi}R_{\rho\sigma\chi\xi}$	$8(D-1)\Lambda^3$	$4(D-1)\Lambda^2$	0	$8\Lambda^2$
$\mathcal{L}_7^{(4)}$	$(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2$	$8D(D-1)\Lambda^3$	$4D(D-1)\Lambda^2$	$8\Lambda^2$	0
$\mathcal{L}_8^{(4)}$	$R^{\mu\nu}R^{\rho\sigma\delta\gamma}R_{\rho}^{\xi}R_{\delta\mu}R_{\sigma\xi\gamma\nu}$	$2(D-1)(D-2)\Lambda^3$	$-\frac{3}{2}(D-1)\Lambda^2$	$\frac{1}{2}\Lambda^2$	$\frac{1}{2}(5D-9)\Lambda^2$
$\mathcal{L}_9^{(4)}$	$R^{\mu\nu}R^{\rho\sigma\delta\gamma}R_{\rho\sigma}^{\xi}R_{\delta\gamma\xi\nu}$	$8(D-1)\Lambda^3$	$6(D-1)\Lambda^2$	0	$6\Lambda^2$
$\mathcal{L}_{10}^{(4)}$	$R^{\mu\nu}R_{\mu}^{\rho}R_{\nu}^{\sigma}R_{\delta\gamma\xi\rho}R^{\delta\gamma\xi}_{\sigma}$	$4(D-1)^2\Lambda^3$	$(D-1)^2\Lambda^2$	$2\Lambda^2$	$(3D-5)\Lambda^2$
$\mathcal{L}_{11}^{(4)}$	$RR_{\mu}^{\rho}R_{\nu}^{\sigma}R_{\rho}^{\delta}R_{\sigma}^{\gamma}R_{\delta}^{\mu}R_{\gamma}^{\nu}$	$2D(D-1)(D-2)\Lambda^3$	$-\frac{3}{2}D(D-1)\Lambda^2$	$\frac{3}{2}(D-2)\Lambda^2$	$\frac{3}{2}D(D-1)\Lambda^2$
$\mathcal{L}_{12}^{(4)}$	$RR_{\mu\nu}^{\rho\sigma}R_{\rho\sigma}^{\delta\gamma}R_{\delta\gamma}^{\mu\nu}$	$8D(D-1)\Lambda^3$	$6D(D-1)\Lambda^2$	$6\Lambda^2$	0
$\mathcal{L}_{13}^{(4)}$	$R^{\mu\nu}R^{\rho\sigma}R_{\mu}^{\delta}R_{\nu}^{\gamma}R_{\rho\delta}R_{\sigma\gamma}$	$4(D-1)^2\Lambda^3$	$(D-1)^2\Lambda^2$	$\frac{1}{2}\Lambda^2$	$\frac{1}{2}(9D-10)\Lambda^2$
$\mathcal{L}_{14}^{(4)}$	$R^{\mu\nu}R^{\rho\sigma}R_{\mu}^{\delta}R_{\nu}^{\gamma}R_{\delta\rho\gamma\sigma}$	$2(D-1)^3\Lambda^3$	0	$\frac{1}{2}(3D-4)\Lambda^2$	$\frac{1}{2}(3D^2-8D+6)\Lambda^2$
$\mathcal{L}_{15}^{(4)}$	$R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho}^{\delta\gamma}R_{\nu\sigma}^{\delta\gamma}$	$4(D-1)^2\Lambda^3$	$(D-1)^2\Lambda^2$	$\Lambda^2$	$(4D-5)\Lambda^2$
$\mathcal{L}_{16}^{(4)}$	$R^{\mu\nu}R_{\nu}^{\rho}R^{\sigma\delta\gamma}_{\mu}R_{\sigma\delta\gamma\rho}$	$4(D-1)^2\Lambda^3$	$(D-1)^2\Lambda^2$	0	$5(D-1)\Lambda^2$
$\mathcal{L}_{17}^{(4)}$	$R_{\delta\gamma}R^{\delta\gamma}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$	$4D(D-1)^2\Lambda^3$	$D(D-1)^2\Lambda^2$	$4(D-1)\Lambda^2$	$D(D-1)\Lambda^2$
$\mathcal{L}_{18}^{(4)}$	$RR_{\mu\nu\rho\sigma}R^{\mu\nu\rho}_{\delta}R^{\sigma\delta}$	$4D(D-1)^2\Lambda^3$	$D(D-1)^2\Lambda^2$	$3(D-1)\Lambda^2$	$2D(D-1)\Lambda^2$
$\mathcal{L}_{19}^{(4)}$	$R^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$	$4D^2(D-1)^2\Lambda^3$	$D^2(D-1)^2\Lambda^2$	$5D(D-1)\Lambda^2$	0
$\mathcal{L}_{20}^{(4)}$	$R^{\mu\nu}R_{\mu\rho\nu\sigma}R^{\delta\rho}R_{\delta}^{\sigma}$	$2(D-1)^3\Lambda^3$	0	$(D-1)\Lambda^2$	$(D-1)(2D-3)\Lambda^2$
$\mathcal{L}_{21}^{(4)}$	$RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma}$	$2D(D-1)^3\Lambda^3$	0	$\frac{1}{2}(D-1)(4D-3)\Lambda^2$	$\frac{1}{2}D(D-1)(2D-3)\Lambda^2$
$\mathcal{L}_{22}^{(4)}$	$R_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\sigma}R_{\sigma}^{\mu}$	$2(D-1)^3\Lambda^3$	0	0	$3(D-1)^2\Lambda^2$
$\mathcal{L}_{23}^{(4)}$	$(R_{\mu\nu}R^{\mu\nu})^2$	$2D(D-1)^3\Lambda^3$	0	$2(D-1)^2\Lambda^2$	$D(D-1)^2\Lambda^2$
$\mathcal{L}_{24}^{(4)}$	$RR_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\mu}$	$2D(D-1)^3\Lambda^3$	0	$\frac{3}{2}(D-1)^2\Lambda^2$	$\frac{3}{2}D(D-1)^2\Lambda^2$
$\mathcal{L}_{25}^{(4)}$	$R^2R_{\mu\nu}R^{\mu\nu}$	$2D^2(D-1)^3\Lambda^3$	0	$\frac{5}{2}D(D-1)^2\Lambda^2$	$\frac{1}{2}D^2(D-1)^2\Lambda^2$
$\mathcal{L}_{26}^{(4)}$	$R^4$	$2D^3(D-1)^3\Lambda^3$	0	$3D^2(D-1)^2\Lambda^2$	0

Table 2: Parameters  $e$ ,  $a$ ,  $b$ ,  $c$  of the linearized equations for all Riemann curvature invariants up to fourth order. We have cross-checked all the terms independently for  $D = 3, 4, 5$  using Mathematica.

which  $m_g^2 = +\infty$ , *i.e.*, those containing a single massless graviton plus an additional spin-0 mode. In terms of the parameters defined in the first section, this condition is  $2a + c = 0$ . Assume now that for certain scalars  $\mathcal{R}_i$  the condition  $2a_i + c_i = 0$  is satisfied for all  $i$ , so that a theory consisting of a linear combination of  $\mathcal{R}_i$  would be free of massive gravitons. From (5.4) we learn that in fact, this property is shared by any theory of the form  $\mathcal{L} = f(\mathcal{R}_1, \dots, \mathcal{R}_m)$  since in that case we find

$$2a + c = \partial_i f(\bar{\mathcal{R}})(2a_i + c_i) = 0. \quad (5.5)$$

Therefore, theories constructed as general functions of scalars whose linear combinations do not produce massive gravitons are also free of those modes. This is a straightforward way of understanding why  $f(R)$ , or more generally  $f(\text{Lovelock})$  theories — see appendix B — inherit the property of Lovelock gravities [25, 26] of not propagating the massive graviton [23, 62].

Something similar happens for theories for which the extra graviton is massless. Assume now that the scalars  $\mathcal{R}_i$  satisfy the condition  $-e_i + 2\Lambda(D-3)a_i = 0$ , so that  $m_g = 0$  for a theory consisting of a linear combination of  $\mathcal{R}_i$ . Then it is straightforward to prove that for a  $f(\mathcal{R}_i)$  theory the mass of the extra graviton is also zero

$$-e + 2\Lambda(D-3)a = \partial_i f(\bar{\mathcal{R}})(-e_i + 2\Lambda(D-3)a_i) = 0. \quad (5.6)$$

Furthermore, note that the condition for the absence of scalar mode reads in turn

$$2a + Dc + 4b(D-1) = \partial_i f(\bar{\mathcal{R}})(2a_i + Dc_i + 4b_i(D-1)) + 4(D-1)\partial_i \partial_j f(\bar{\mathcal{R}})e_i e_j = 0. \quad (5.7)$$

This expression is more complicated than (5.5) since the expression for  $b$  in (5.4) contains a term involving the  $e_i$ . This is not surprising:  $f(R)$  does propagate the additional scalar mode even though Einstein gravity does not.

## 6 Einsteinian quartic gravities

In [29], we constructed a cubic theory which only propagates a massless graviton on msb. The theory was defined in a dimension-independent way, in the sense that the relative couplings between the different invariants involved in its definition were the same in all dimensions. In fact, we proved that up to cubic order in curvature, the most general theory satisfying those requirements reads

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} \alpha \mathcal{X}_4 + \kappa^{\frac{6-D}{D-2}} [\beta \mathcal{X}_6 + \lambda \mathcal{P}] \right\}, \quad (6.1)$$

where  $\mathcal{X}_4$  and  $\mathcal{X}_6$  are respectively the dimensionally-extended Euler densities for  $D = 4$  and  $D = 6$  manifolds.  $\mathcal{X}_4$  is defined below (2.24) and  $\mathcal{X}_6$  is given in (B.22). Hence,



the only terms appearing in (6.1) are the Lovelock ones plus the new *Einsteinian cubic gravity* term  $\mathcal{P}$ , defined as

$$\mathcal{P} \equiv 12R_{\mu}^{\rho} R_{\nu}^{\sigma} R_{\rho}^{\gamma} R_{\sigma}^{\delta} R_{\gamma}^{\mu} R_{\delta}^{\nu} + R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\gamma\delta} R_{\gamma\delta}^{\mu\nu} - 12R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + 8R_{\mu}^{\nu} R_{\nu}^{\rho} R_{\rho}^{\mu}. \quad (6.2)$$

The effective Einstein constant for the ECG theory (6.1) is

$$\kappa_{\text{eff}} = \kappa \left[ 1 + 4\kappa^{\frac{2}{D-2}} \Lambda \alpha (D-4)(D-3) + 6\kappa^{\frac{4}{D-2}} \Lambda^2 (D-6)(D-3)((D-5)(D-4)\beta - 4\lambda) \right]^{-1}. \quad (6.3)$$

Interestingly, when restricted to  $D = 4$ , the above theory reduces to

$$S = \int_{\mathcal{M}} d^4x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa \lambda \mathcal{P} \right\}, \quad (6.4)$$

given that in that number of dimensions  $\mathcal{X}_4$  is topological and  $\mathcal{X}_6$  vanishes identically.

In this section we will explain how to extend the above construction to quartic theories. We will take advantage of the results in section 4 to construct *Einsteinian quartic gravities*.

As we have just reviewed, the construction of *Einsteinian gravities* requires the theories to be defined in a dimension-independent fashion. Apart from esthetics, there are some practical reasons to consider theories satisfying this property. Firstly, observe that this property is shared by all Lovelock gravities, which are the most general metric theories of gravity with divergence-free second-order equations of motion — at the full non-linear level — in any number of dimensions [25, 26].

In addition, theories defined in this way have the nice feature that they preserve the total number of dof under compactification, in the following sense. Consider for example the Kaluza-Klein reduction of the  $D$ -dimensional EH term along some direction  $x^0$ . The metric  $g_{MN}$ , which propagates  $D(D-3)/2$  dof, gives rise to a  $(D-1)$ -dimensional metric  $g_{\mu\nu}$  which contains  $(D-1)(D-4)/2$  dof, plus a 1-form  $A_{\mu} \equiv g_{\mu 0}$  with  $(D-3)$  dof and a scalar field  $\phi \equiv g_{00}$  with 1 dof. This property is shared by Einsteinian gravities, but not by theories which have a dimension-dependent definition. If a theory of that kind only propagates the  $D(D-3)/2$  dof of the massless graviton in  $D$  dimensions, it will give rise to extra degrees of freedom when compactified, because the lower-dimensional metric will in general propagate the extra spin-2 and scalar modes in addition to the  $(D-1)(D-4)/2 + (D-3) + 1 = D(D-3)/2$  dof of the massless graviton, the 1-form and the scalar. From a similar perspective, if we consider some  $D$ -dimensional theory and assume some of the dimensions of our spacetime to be compact, *e.g.*,  $\mathcal{M}^D = \mathcal{M}_{\text{nc}}^{D'} \times \mathcal{M}_{\text{c}}^{D-D'}$ , where  $\mathcal{M}_{\text{c}}^{D-D'}$  is some compact manifold, then the resulting effective action on the non-compact dimensions will involve the same gravitational term only if this has been defined in a dimension-independent fashion — see *e.g.*, [64, 65] for the

Kaluza-Klein reduction of Gauss-Bonnet gravity. This is exactly what happens with the Einstein-Hilbert term in general String Theory compactifications<sup>18</sup>.

As explained in previous sections, the constraints required for a theory to share the spectrum of Einstein gravity at the linearized level can be written as  $2a + c = 4b + c = 0$ , which account for the conditions  $m_g^2 = m_s^2 = +\infty$ . Imposing those conditions at each order in curvature for the theory (4.2), one is left with six constraints on the coupling values,  $F_g^{(2)}(\alpha_i) = F_s^{(2)}(\alpha_i) = F_g^{(3)}(\beta_i, D) = F_s^{(3)}(\beta_i, D) = F_g^{(4)}(\gamma_i, D) = F_s^{(4)}(\gamma_i, D) = 0$  — see appendix D for the explicit expressions. If these constraints are satisfied, the theory will only propagate a massless graviton on a msb. Imposing each constraint to be satisfied independently of the dimension multiplies the number of constraints. This is because *e.g.*,  $F_{g,s}^{(3)}(\beta_i, D)$  is a polynomial of degree 2 in  $D$ , so we need to impose the coefficients of the  $D^0$ ,  $D^1$  and  $D^2$  terms to vanish independently. More generally, at  $n$ -th order in curvature, the corresponding constraints are polynomials of degree  $2n - 4$  in  $D$ , and hence we will find  $2n - 3$  constraints coming from the absence of the massive graviton, and the same number from imposing the absence of scalar, which makes  $2(2n - 3)$  in total. At the quartic level this means 10 constraints. Since in general dimensions there are up to 26 independent invariants at this order in curvature [63] — see Table 2, this means that there exists a 16-parameter family of Einsteinian quartic gravities. If we choose the 16 parameters to be  $\{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6, \gamma_7, \gamma_8, \gamma_9, \gamma_{10}, \gamma_{12}, \gamma_{13}, \gamma_{14}, \gamma_{18}, \gamma_{20}, \gamma_{26}\}$ , the rest of couplings are given in terms of these as

$$\begin{aligned}\gamma_{11} &= +\frac{1}{3}(12\gamma_{12} - 4\gamma_1 + 12\gamma_2 - 8\gamma_3 + 36\gamma_4 + 72\gamma_5 + 16\gamma_6 + 16\gamma_7 - 3\gamma_8 + 12\gamma_9), \quad (6.5) \\ \gamma_{15} &= +\frac{1}{2}(-10\gamma_1 - 4\gamma_{10} - \gamma_{13} + \gamma_{14} + 16\gamma_2 - 14\gamma_3 + 48\gamma_4 + 96\gamma_5 + 16\gamma_6 - 4\gamma_8 + 12\gamma_9), \\ \gamma_{16} &= +\frac{1}{10}(36\gamma_1 + 10\gamma_{10} - 24\gamma_{12} - 5\gamma_{13} - 5\gamma_{14} - 74\gamma_2 - 2\gamma_{20} + 1140\gamma_{26} + 57\gamma_3 - 210\gamma_4 \\ &\quad - 420\gamma_5 - 84\gamma_6 - 20\gamma_7 + 17\gamma_8 - 72\gamma_9), \\ \gamma_{17} &= -\gamma_{18} - 120\gamma_{26}, \\ \gamma_{19} &= +6\gamma_{26},\end{aligned}$$

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<sup>18</sup>For example, the 10-dimensional type-IIA String Theory effective action reduces to a class of  $D = 4$ ,  $\mathcal{N} = 2$  Supergravity theories when 6 of the dimensions are compact on a Calabi-Yau threefold — see *e.g.*, [66]. In the type-IIA action, the leading contribution from the metric is the 10-dimensional Einstein-Hilbert term  $R^{(10)}$ . Under compactification, this produces  $R^{(4)}$  — plus additional terms involving other fields.

$$\begin{aligned}
\gamma_{21} &= +8\gamma_1 - 12\gamma_{12} - 3\gamma_{14} + 2\gamma_{18} - 18\gamma_2 - 2\gamma_{20} + 900\gamma_{26} + 13\gamma_3 - 54\gamma_4 - 108\gamma_5 - 20\gamma_6 \\
&\quad - 20\gamma_7 + 3\gamma_8 - 12\gamma_9, \\
\gamma_{22} &= +\frac{1}{10}(16\gamma_1 - 24\gamma_{12} - 10\gamma_{14} - 14\gamma_2 - 2\gamma_{20} + 1140\gamma_{26} + 17\gamma_3 - 50\gamma_4 - 100\gamma_5 - 4\gamma_6 \\
&\quad - 20\gamma_7 + 2\gamma_8 + 8\gamma_9), \\
\gamma_{23} &= +\frac{1}{20}(-154\gamma_1 + 216\gamma_{12} + 60\gamma_{14} - 40\gamma_{18} + 306\gamma_2 + 38\gamma_{20} - 22260\gamma_{26} - 233\gamma_3 \\
&\quad + 930\gamma_4 + 1860\gamma_5 + 316\gamma_6 + 340\gamma_7 - 48\gamma_8 + 168\gamma_9), \\
\gamma_{24} &= +\frac{1}{30}(-6\gamma_1 + 24\gamma_{12} + 54\gamma_2 + 2\gamma_{20} + 9060\gamma_{26} - 27\gamma_3 + 150\gamma_4 + 300\gamma_5 + 84\gamma_6 \\
&\quad + 60\gamma_7 - 12\gamma_8 + 72\gamma_9), \\
\gamma_{25} &= -24\gamma_{26}.
\end{aligned} \tag{6.6}$$

Plugging these back in the original quartic action, we obtain the family of 16 independent Einsteinian quartic gravities. In four dimensions, it can be seen that only 13 of the 26 invariants in Table 2 are non-vanishing and independent from each other [63]. In particular, we can set  $\gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_6 = \gamma_8 = \gamma_9 = \gamma_{10} = \gamma_{12} = \gamma_{13} = \gamma_{14} = \gamma_{18} = \gamma_{20} = 0$  — the choice being non-unique. This leaves us with a three-parameter family of independent four-dimensional Einsteinian quartic gravities. More explicitly, (6.6) becomes now

$$\begin{aligned}
\gamma_{11} &= +8/3(9\gamma_5 + 2\gamma_7), & \gamma_{15} &= +48\gamma_5, \\
\gamma_{16} &= +114\gamma_{26} - 42\gamma_5 - 2\gamma_7, & \gamma_{17} &= -120\gamma_{26}, \\
\gamma_{19} &= +6\gamma_{26}, & \gamma_{21} &= +4(225\gamma_{26} - 27\gamma_5 - 5\gamma_7), \\
\gamma_{22} &= +2(57\gamma_{26} - 5\gamma_5 - \gamma_7), & \gamma_{23} &= -1113\gamma_{26} + 93\gamma_5 + 17\gamma_7, \\
\gamma_{24} &= +2(151\gamma_{26} + 5\gamma_5 + \gamma_7), & \gamma_{25} &= -24\gamma_{26}.
\end{aligned} \tag{6.7}$$

where the three parameters are  $\{\gamma_5, \gamma_7, \gamma_{26}\}$ . Using these relations we have constructed the following basis of invariants

$$\begin{aligned}
\mathcal{Q}_1 &\equiv +3R^{\mu\nu\rho\sigma}R_{\mu\nu}^{\gamma\delta}R_{\gamma\delta}^{\alpha\beta}R_{\rho\sigma\alpha\beta} - 15(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 - 8RR_{\mu}^{\rho}{}_{\nu}{}^{\sigma}R_{\rho}^{\gamma}{}_{\sigma}{}^{\delta}R_{\gamma}^{\mu}{}_{\delta}{}^{\nu} \\
&\quad + 144R^{\mu\nu}R^{\rho\sigma}R_{\mu\rho}^{\gamma\delta}R_{\gamma\delta\nu\sigma} - 96R^{\mu\nu}R_{\nu}^{\rho}R^{\alpha\beta\gamma}{}_{\mu}R_{\alpha\beta\gamma\rho} - 24RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} \\
&\quad + 24(R_{\mu\nu}R^{\mu\nu})^2, \\
\mathcal{Q}_2 &\equiv +3(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 + 16RR_{\mu}^{\rho}{}_{\nu}{}^{\sigma}R_{\rho}^{\gamma}{}_{\sigma}{}^{\delta}R_{\gamma}^{\mu}{}_{\delta}{}^{\nu} - 6R^{\mu\nu}R_{\nu}^{\rho}R^{\alpha\beta\gamma}{}_{\mu}R_{\alpha\beta\gamma\rho} \\
&\quad - 60RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} - 6R_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\sigma}R_{\sigma}^{\mu} + 51(R_{\mu\nu}R^{\mu\nu})^2 + 6RR_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\mu}, \\
\mathcal{Q}_3 &\equiv +R^4 + 57(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma})^2 - 120R_{\gamma\delta}R^{\gamma\delta}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} + 6R^2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \\
&\quad - 240RR_{\mu\nu\rho\sigma}R^{\mu\rho}R^{\nu\sigma} - 144(R_{\mu\nu}R^{\mu\nu})^2 + 416RR_{\mu}^{\nu}R_{\nu}^{\rho}R_{\rho}^{\mu} - 24R^2R_{\mu\nu}R^{\mu\nu} \\
&\quad + 304RR_{\mu}^{\rho}{}_{\nu}{}^{\sigma}R_{\rho}^{\delta}{}_{\sigma}{}^{\gamma}R_{\delta}^{\mu}{}_{\gamma}{}^{\nu}.
\end{aligned} \tag{6.8}$$

Just like its cubic cousin  $\mathcal{P}$  defined in (6.2),  $\mathcal{Q}_1$ ,  $\mathcal{Q}_2$  and  $\mathcal{Q}_3$  — or any linear combination of them — only propagate the usual massless graviton when linearized on a msb, not only in  $D = 4$ , but in any number of dimensions<sup>19</sup>.

## 7 New ghost-free gravity

In the previous section we reviewed ECG, and extended the construction to quartic theories. As we explained, all those theories are free both of the ghost-like graviton and the scalar mode on a msb. In this section we will relax the second condition to construct the most general cubic theory defined in a dimension-independent manner which does not propagate the massive graviton — but does in general include the scalar. As far as we know, the most general known theories which satisfy these requirements are those defined as functions of Lagrangian densities which, when considered as theories by themselves, do not propagate the massive graviton — a property firstly proven in section 5. All the known examples reduce to  $f$ (Lovelock) gravities and the more exotic case of  $f$ (ECG) or functions of the quartic theories studied in the previous section.

Recall that the condition for the absence of massive gravitons is  $2a + c = 0$ . If we impose this on the general theory defined in (4.2) up to cubic order and ask it to be satisfied independently of the spacetime dimension, we are left with the conditions

$$\frac{1}{2}\alpha_2 + 2\alpha_3 = 0, \quad (7.1)$$

$$-\beta_1 + 8\beta_2 - \beta_5 - \beta_6 = 0, \quad (7.2)$$

$$2\beta_3 - 2\beta_4 + \beta_5 + \frac{3}{2}\beta_6 - \frac{1}{2}\beta_7 = 0, \quad (7.3)$$

$$2\beta_4 + \frac{1}{2}\beta_7 = 0. \quad (7.4)$$

Hence, there are two independent quadratic terms and five cubic ones. They can all be written as

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (R - 2\Lambda_0) + \kappa^{\frac{4-D}{D-2}} (\tilde{\alpha}_1 R^2 + \tilde{\alpha}_2 \mathcal{X}_4) + \kappa^{\frac{6-D}{D-2}} (\tilde{\beta}_1 R^3 + \tilde{\beta}_2 \mathcal{X}_6 + \tilde{\beta}_3 R \mathcal{X}_4 + \tilde{\beta}_4 \mathcal{P} + \tilde{\beta}_5 \mathcal{Y}) \right\}. \quad (7.5)$$

In this action we find all the  $f$ (Lovelock) terms up to this order in curvature, as well as two additional theories. The first,  $\mathcal{P}$ , is nothing but the Einsteinian cubic term defined in (6.2), while the second is a previously unidentified invariant which reads

$$\mathcal{Y} \equiv R_{\mu}^{\alpha} R_{\nu}^{\beta} R_{\alpha}^{\rho} R_{\beta}^{\sigma} R_{\rho}^{\mu} R_{\sigma}^{\nu} - 3R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + 2R_{\mu}^{\nu} R_{\nu}^{\rho} R_{\rho}^{\mu}. \quad (7.6)$$

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<sup>19</sup>We have cross-checked the linearized equations of  $\mathcal{P}$  and  $\mathcal{Q}_i$ ,  $i = 1, 2, 3$  for  $D = 4, 5, 6$  using the Mathematica package xAct [67].

In the above expression, the pure Lovelock terms,  $R$ ,  $\mathcal{X}_4$  and  $\mathcal{X}_6$  as well as  $\mathcal{P}$  do not contribute to the denominator of the scalar mass — and hence any linear combination of those terms alone would yield  $m_s^2 = +\infty$  — while  $R^2$ ,  $R^3$ ,  $R\mathcal{X}_4$  and  $\mathcal{Y}$  do. Indeed, we obtain for this *New ghost-free gravity*<sup>20</sup> (7.5)

$$\begin{aligned}
m_s^2 = & \left[ D - 2 + 4(D - 4)\kappa^{\frac{2}{D-2}}\Lambda \left( \tilde{\alpha}_1(D - 1)D + \tilde{\alpha}_2(D - 3)(D - 2) \right) \right. \\
& + 6(D - 6)\kappa^{\frac{4}{D-2}}\Lambda^2 \left( \tilde{\beta}_1(D - 1)^2 D^2 + \tilde{\beta}_2(D - 5)(D - 4)(D - 3)(D - 2) \right. \\
& \left. \left. + \tilde{\beta}_3(D - 3)(D - 2)(D - 1)D - 4\tilde{\beta}_4(D - 3)(D - 2) - \tilde{\beta}_5(D(D - 3) + 3) \right) \right] \\
& \times \left[ 8(D - 1) \left( \kappa^{\frac{2}{D-2}}\tilde{\alpha}_1 + \kappa^{\frac{4}{D-2}}\Lambda \left( 3\tilde{\beta}_1(D - 1)D + 2\tilde{\beta}_3(D - 3)(D - 2) - 3/2\tilde{\beta}_5 \right) \right) \right]^{-1}.
\end{aligned} \tag{7.7}$$

Hence, setting  $\tilde{\alpha}_1 = \tilde{\beta}_1 = \tilde{\beta}_3 = \tilde{\beta}_5 = 0$ , one finds  $m_s^2 = +\infty$ , as expected. It is also worth pointing out that, just like ECG,  $\mathcal{Y}$  is non-trivial in four-dimensions. Moreover, the effective gravitational constant reads now

$$\begin{aligned}
\kappa_{\text{eff}} = & \kappa \left[ 1 + 4\kappa^{\frac{2}{D-2}}\Lambda \left( \tilde{\alpha}_1(D - 1)D + \tilde{\alpha}_2(D - 4)(D - 3) \right) \right. \\
& + 6\kappa^{\frac{4}{D-2}}\Lambda^2 \left( \tilde{\beta}_1(D - 1)^2 D^2 + \tilde{\beta}_2(D - 6)(D - 5)(D - 4)(D - 3) \right. \\
& \left. \left. + \tilde{\beta}_3(D - 10/3)(D - 3)(D - 1)D - 4\tilde{\beta}_4(D - 6)(D - 3) - \tilde{\beta}_5((D - 5)D + 9) \right) \right]^{-1}.
\end{aligned} \tag{7.8}$$

An interesting property of  $f(\text{Lovelock})$  gravities is that they are ghost-free at the full non-linear level, since they can be written as scalar-Lovelock theories with second-order equations of motion [62, 68]. It is natural to wonder if  $\mathcal{Y}$  has any chance of sharing this property. More generally, it would be interesting to explore further properties of this new cubic term.

## 8 Generalized Newton potential

In this section we use the results of section 2 to compute the Newton potential  $U_D(r)$  and the *Parametrized Post-Newtonian* (PPN) parameter  $\gamma$  for a general theory of the form (1.1) in general dimensions. We start reviewing the four-dimensional case and then we extend our results to arbitrary  $D$ , pointing out interesting differences with respect to the  $D = 4$  case<sup>21</sup>.

<sup>20</sup>We stress that we have only proven this theory to be free of ghost modes at the linearized level.

<sup>21</sup>Throughout this section and the following we will tacitly assume that  $m_s^2, m_g^2 \geq 0$ .

## 8.1 Four dimensions

The analysis performed in section 2.3.2 tells us that in order to obtain a solution of the linearized equations in a flat background we must solve equations (2.48), (2.52) and (2.53), and then reconstruct the metric perturbation (2.51)<sup>22</sup>.

If we denote by  $H_{\mu\nu}(x; m)$  the general solution of the Klein-Gordon equation

$$(\square - m^2) H_{\mu\nu}(x; m) = -4\pi T_{\mu\nu}(x), \quad (8.1)$$

and by  $H(x; m)$  its trace, the solutions to (2.48), (2.52) and (2.53) can be written as

$$\hat{h}_{\mu\nu} = \frac{\kappa_{\text{eff}}}{2\pi} H_{\mu\nu}(0), \quad \phi = \frac{\kappa_{\text{eff}}}{2\pi} H(m_s), \quad t_{\mu\nu} = -\frac{\kappa_{\text{eff}}}{2\pi} \left[ H_{\langle\mu\nu\rangle}(m_g) + \frac{1}{3m_g^2} \partial_{\langle\mu} \partial_{\nu\rangle} H(m_g) \right]. \quad (8.2)$$

Inserting this into the metric perturbation (2.51) and making the gauge transformation

$$h_{\mu\nu}^N \equiv h_{\mu\nu} - \partial_{(\mu} \xi_{\nu)}, \quad (8.3)$$

where  $N$  stands for “Newtonian gauge” and

$$\xi_\nu \equiv \frac{1}{3} \partial_\nu \left( (m_g^{-2} - m_s^{-2}) H(0) + m_s^{-2} H(m_s) - m_g^{-2} H(m_g) \right), \quad (8.4)$$

we obtain after some simplifications

$$h_{\mu\nu}^N = \frac{\kappa_{\text{eff}}}{8\pi} \left[ 4H_{\mu\nu}(0) - 4H_{\mu\nu}(m_g) + \eta_{\mu\nu} \left( -2H(0) + \frac{4}{3}H(m_g) + \frac{2}{3}H(m_s) \right) \right]. \quad (8.5)$$

Now if we restrict ourselves to static configurations, (8.1) reduces to the so-called screened Poisson equation,  $(\Delta - m^2) H_{\mu\nu}(\vec{x}; m) = -4\pi T_{\mu\nu}(\vec{x})$ , whose general solution reads

$$H_{\mu\nu}(\vec{x}; m) = \int d^3 \vec{x}' \frac{T_{\mu\nu}(\vec{x}')}{|\vec{x} - \vec{x}'|} e^{-m|\vec{x} - \vec{x}'|}. \quad (8.6)$$

This can be seen as a superposition of functions  $1/|\vec{x} - \vec{x}'|$  weighted by the source  $T_{\mu\nu}(\vec{x}')$  and with an exponential screening controlled by the mass  $m$ . Using this we can rewrite (8.5) as

$$h_{\mu\nu}^N(x) = \frac{\kappa_{\text{eff}}}{8\pi} \int d^3 \vec{x}' T_{\alpha\beta}(\vec{x}') \Pi^{\alpha\beta}_{\mu\nu}(\vec{x} - \vec{x}'), \quad (8.7)$$

---

<sup>22</sup>The same procedure can be naturally carried out for an (A)dS background using the expressions in section 2.3.1. We find that the results are approximately the same provided we consider distances shorter than the (A)dS scale  $r \ll |\Lambda|^{-1/2}$  and  $m_g^2 \gg |\Lambda|$ . This is useful because in the flat case one cannot easily set the masses  $m_g$  and  $m_s$  to zero as only the Einstein-Hilbert term contributes to the numerator of those quantities when  $\Lambda = 0$  — see *e.g.*, (2.32)-(2.34). In the (A)dS case, terms of all orders contribute and it is in principle possible to set  $m_s = 0$  or  $m_g = 0$ .

where the static propagator reads

$$\Pi^{\alpha\beta}_{\mu\nu}(\vec{x} - \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} \left[ 4\delta^\alpha_{(\mu}\delta^\beta_{\nu)} \left( 1 - e^{-m_g|\vec{x}-\vec{x}'|} \right) - 2\eta^{\alpha\beta}\eta_{\mu\nu} \left( 1 - \frac{2}{3}e^{-m_g|\vec{x}-\vec{x}'|} - \frac{1}{3}e^{-m_s|\vec{x}-\vec{x}'|} \right) \right]. \quad (8.8)$$

Now, let us apply the previous expressions to the case of a solid and static sphere of radius  $R$  and mass  $M$  on a flat background. For this distribution of matter, the only non-vanishing component of the stress-tensor reads

$$T_{00}(r) = \rho(r) = \rho_0 \theta(R - r), \quad \text{with} \quad \rho_0 \equiv \frac{M}{4\pi R^3/3}, \quad (8.9)$$

where  $\theta(x)$  is the Heaviside step function. For this configuration the result for  $H_{00}(r; m) = -H(r; m)$  in the outer region  $r > R$  obtained from (8.6) reads

$$H(r; m) = -f(mR) \frac{M}{r} e^{-mr}, \quad (8.10)$$

where  $f(mR)$  is a form factor given by

$$f(mR) = \frac{3}{(mR)^3} \left[ mR \cosh(mR) - \sinh(mR) \right], \quad (8.11)$$

which behaves as  $f(mR) \approx \frac{3}{2} \frac{1}{(mR)^2} e^{mR}$  if  $mR \gg 1$  and as  $f(mR) \approx 1$  in the point-like limit, *i.e.*, when  $mR \ll 1$ . Finally, inserting these results into the metric  $h^N_{\mu\nu}$  in (8.5) and this in  $g^N_{\mu\nu} = \eta_{\mu\nu} + h^N_{\mu\nu}$  we obtain

$$ds_N^2 = -(1 + 2U(r))dt^2 + (1 - 2\gamma(r)U(r))\delta_{ij}dx^i dx^j, \quad (8.12)$$

where  $U(r)$  and  $\gamma(r)$  are given by

$$U(r) = -\frac{G_{\text{eff}}M}{r} \left[ 1 - \frac{4}{3}f(m_g R)e^{-m_g r} + \frac{1}{3}f(m_s R)e^{-m_s r} \right], \quad (8.13)$$

$$\gamma(r) = \frac{3 - 2f(m_g R)e^{-m_g r} - f(m_s R)e^{-m_s r}}{3 - 4f(m_g R)e^{-m_g r} + f(m_s R)e^{-m_s r}}, \quad (8.14)$$

and  $G_{\text{eff}} \equiv \kappa_{\text{eff}}/(8\pi)$ . Evaluating these expressions in the point-like limit of the sphere  $f(mR) = 1$  we finally obtain the generalized Newtonian potential and the PPN parameter  $\gamma$

$$U(r) = -\frac{G_{\text{eff}}M}{r} \left[ 1 - \frac{4}{3}e^{-m_g r} + \frac{1}{3}e^{-m_s r} \right], \quad \gamma(r) = \frac{3 - 2e^{-m_g r} - e^{-m_s r}}{3 - 4e^{-m_g r} + e^{-m_s r}}. \quad (8.15)$$

Let us make some comments about these results. First, observe that the usual Newton potential gets corrected by two Yukawa-like terms controlled by the masses of the two extra modes which can be computed for a given theory through (2.29) and (2.30). The above expression for  $U(r)$  has been obtained before using different methods — see *e.g.*, [30, 31, 69]<sup>23</sup>. Note that while the contribution from the scalar has the usual sign for a Yukawa potential, the massive graviton one comes with the opposite sign, which is another manifestation of its ghost nature. Observe also that the whole contribution from the higher-derivative terms appears through  $m_g$  and  $m_s$ , the coefficients  $-4/3$  and  $1/3$  in front of the exponentials being common to all theories. In table 3 we present

	$U(r)/G_{\text{eff}}$	$\gamma$
$m_s = m_g = +\infty$	$-M/r$	1
$m_s = +\infty,  m_g r  \ll 1$	$+M/(3r)$	-1
$m_s = 0, m_g = +\infty$	$-4M/(3r)$	1/2
$m \equiv m_g = m_s$	$-M(1 - e^{-mr})/r$	1

Table 3: Newton’s potential and  $\gamma(r)$  for various values of the masses of the extra modes.

the values of  $U(r)$  and  $\gamma$  for different limiting values of  $m_s$  and  $m_g$ . Naturally, when  $m_g, m_s \gg 1$  one is left with the Einsteinian values of the Newton potential and  $\gamma$ , and the same happens if we go sufficiently far away from  $M$  for arbitrary values of the extra mode masses. It is also interesting that the only cases for which the potential is divergent as  $r \rightarrow 0$  are those for which at least one of the extra modes is absent, *i.e.*, when either  $m_s = +\infty$ , or  $m_g = +\infty$  or both  $m_g = m_s = +\infty$ .

Indeed,  $U(r)$  does not diverge as  $r \rightarrow 0$  in the general case. In fact, one finds

$$U(r) = -G_{\text{eff}} M \left[ \frac{(4m_g - m_s)}{3} - \frac{(4m_g^2 - m_s^2)r}{6} + \mathcal{O}(r^2) \right], \quad (8.16)$$

which is a negative constant at  $r = 0$  when  $m_g > m_s/4$  (and viceversa). The potential grows linearly with  $r$  at first order for  $m_g > m_s/2$  and in that case it is monotonous in the whole range of  $r$ . When  $m_g < m_s/2$  instead,  $U(r)$  decreases linearly near  $r = 0$  and it has a minimum at some intermediate value of  $r$ . Plots of  $U(r)/G_{\text{eff}}$  for various values of the masses satisfying the different situations can be found in Fig. 1.

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<sup>23</sup>See *e.g.*, [70, 71] for results corresponding to higher-order gravities involving covariant derivatives of the Riemann tensor.



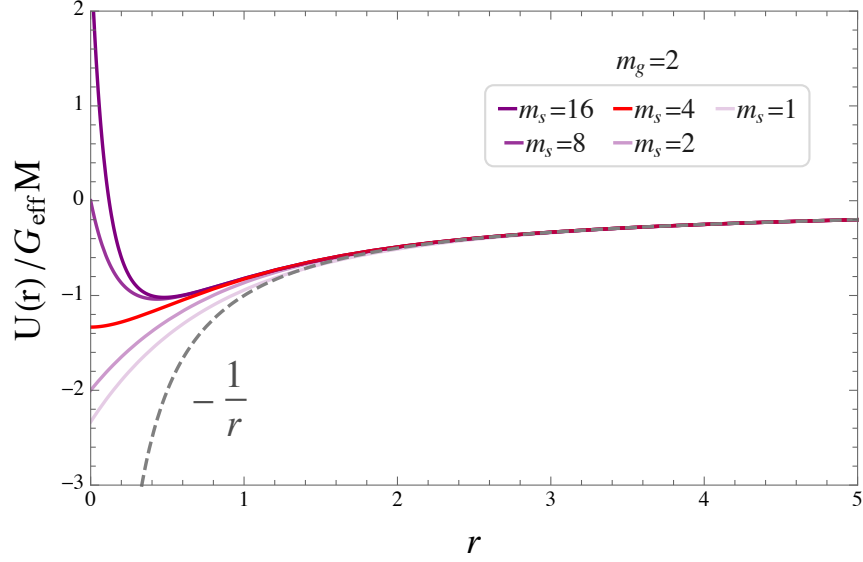


Figure 1:  $U(r)/(G_{\text{eff}}M)$  for  $m_g = 2$  and  $m_s = 16, 8, 2, 1$  (purple curves), and  $m_s = 4$  (red) and the usual Newton potential (dashed gray).

## 8.2 Higher dimensions

The analysis of the previous section can be extended to general dimensions  $D \geq 4$ . The metric perturbation in the Newtonian gauge can be seen to be given by

$$h_{\mu\nu}^N = 4G_{\text{eff}} \left[ H_{\mu\nu}(0) - H_{\mu\nu}(m_g) + \frac{\eta_{\mu\nu}}{(D-1)(D-2)} (-(D-1)H(0) + (D-2)H(m_g) + H(m_s)) \right], \quad (8.17)$$

where again  $H_{\mu\nu}(m)$  is a solution of (8.1). In the static case, we can write the solution explicitly as

$$H_{\mu\nu}(\vec{x}; m) = 2 \left( \frac{m}{2\pi} \right)^{\frac{D-3}{2}} \int d^{D-1} \vec{x}' \frac{T_{\mu\nu}(\vec{x}')}{|\vec{x} - \vec{x}'|^{\frac{D-3}{2}}} K_{\frac{D-3}{2}}(m|\vec{x} - \vec{x}'|), \quad (8.18)$$

where  $K_\ell(x)$  is the modified Bessel function of the second kind. Now, specializing to a static point-like particle of mass  $M$ , we can obtain the  $D$ -dimensional version of (8.12).

The Newtonian potential and the gamma parameter read, respectively,

$$U_D(r) = -\mu(D) \frac{G_{\text{eff}} M}{r^{D-3}} \left[ 1 + \nu(D) r^{\frac{D-3}{2}} \left[ -m_g^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(m_g r) + \frac{m_s^{\frac{D-3}{2}}}{(D-2)^2} K_{\frac{D-3}{2}}(m_s r) \right] \right],$$

$$\gamma_D(r) = \frac{1 - \frac{2}{(D-1)\Gamma(\frac{D-3}{2})} \left[ (D-2) \left( \frac{m_g r}{2} \right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(m_g r) + \left( \frac{m_s r}{2} \right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(m_s r) \right]}{D-3 - \frac{2}{(D-1)\Gamma(\frac{D-3}{2})} \left[ (D-2)^2 \left( \frac{m_g r}{2} \right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(m_g r) - \left( \frac{m_s r}{2} \right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(m_s r) \right]}, \quad (8.19)$$

with

$$\mu(D) \equiv \frac{8\pi}{(D-2)\Omega_{D-2}}, \quad \text{and} \quad \nu(D) \equiv \frac{(D-2)^2}{\Gamma\left[\frac{D+1}{2}\right] 2^{\frac{D-1}{2}}}, \quad (8.20)$$

and where  $\Omega_{D-2} \equiv 2\pi^{\frac{D-1}{2}}/\Gamma[\frac{D-1}{2}]$  is the volume of the  $(D-2)$ -dimensional unit sphere. When  $2\ell$  is odd, *i.e.*, for even  $D$ , the Bessel functions  $K_\ell(x)$  can be written explicitly in terms of elementary functions as

$$K_{\frac{D-3}{2}}(x) = e^{-x} \sqrt{\frac{\pi}{2x}} \sum_{j=1}^{\frac{D-2}{2}} \frac{(D-3-j)!}{(j-1)!(\frac{D-2}{2}-j)!(2x)^{\frac{D-2}{2}-j}}, \quad (\text{even } D) \quad (8.21)$$

which allows for a simplification of (8.19) in those cases, and from which it is easy to reproduce the  $D=4$  results (8.15) presented in the previous section. From (8.19) we infer that the usual four-dimensional Yukawa potential for a force-mediating particle of mass  $m$  generalizes to higher dimensions as

$$U_{D,\text{Yukawa}}(r) \sim \left(\frac{m}{r}\right)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(mr). \quad (8.22)$$

Going back to higher-order gravities, observe that close to the origin, the generalized Newton potential  $U_D(r)$  behaves for  $D > 5$  as

$$U_D(r \rightarrow 0) \sim -\frac{G_{\text{eff}} M [(D-2)^2 m_g^2 - m_s^2]}{r^{D-5}} + \dots, \quad (8.23)$$

up to a positive dimension-dependent constant for generic values of  $m_g$  and  $m_s$ . For  $D=4$  we find a constant term (8.16), while for  $D=5$  one finds a logarithmic divergence instead

$$U_5(r \rightarrow 0) = \frac{G_{\text{eff}} M}{12\pi} (9m_g^2 - m_s^2) \log r + \mathcal{O}(r^0). \quad (8.24)$$

This means that for generic values of the extra mode masses,  $U_D(r)$  is divergent at  $r=0$  in all dimensions higher than four. In Fig. 2 we plot  $U_5(r)$ , which can be explicitly written as

$$U_5(r) = -\frac{G_{\text{eff}} M}{6\pi r^2} [8 - 9m_g r K_1(m_g r) + m_s r K_1(m_s r)]. \quad (8.25)$$

As expected, most curves in Fig. 2 diverge at the origin. There is an exception (and only

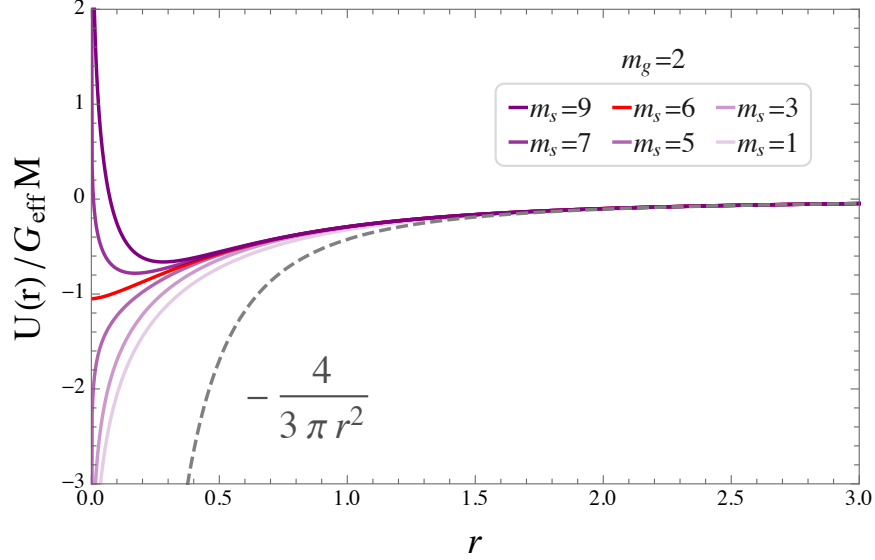


Figure 2:  $U(r)/(G_{\text{eff}}M)$  in  $D = 5$  for  $m_g = 2$  and  $m_s = 1, 3, 5, 7, 9$  (purple curves), and  $m_s = 6$  (red) and the usual Newton potential in five dimensions (dashed gray).

one) though, which corresponds to the case  $m_g = m_s/3$ , for which the potential is finite everywhere. The value  $m_g = \frac{m_s}{(D-2)}$  is special in general dimensions, as it determines the transition between two kinds of potentials. In particular, when  $m_g > \frac{m_s}{(D-2)}$ ,  $U_D(r)$  is monotonous in the whole range of  $r$  and diverges to  $-\infty$  at the origin, while for  $m_g < \frac{m_s}{(D-2)}$  it has a minimum at some finite value of  $r$  and  $U_D(r \rightarrow 0) \rightarrow +\infty$  instead — see Fig. 2 for an illustration of these features in the five-dimensional case. For the particular value  $m_g = \frac{m_s}{(D-2)}$ , the potential is also finite at the origin for  $D = 6$ , but not for  $D \geq 7$ .

In table 4 we present some particular cases for  $U_D(r)$  and  $\gamma_D$ <sup>24</sup> corresponding to different limiting values of  $m_g$  and  $m_s$ . Once again, when  $m_g, m_s \gg 1$ , one is left with the Einsteinian values of the corresponding Newton potentials and  $\gamma_D$ , and the same happens at sufficiently large distances from  $M$  for general values of the extra mode masses. Just like in four dimensions, when the masses of the extra modes are equal,  $m_s = m_g$ , the gamma parameter coincides with that of Einstein gravity,  $\gamma_D = 1/(D-3)$ .

<sup>24</sup>We use the following two limits of the modified Bessel functions:

$$\lim_{x \rightarrow \infty} x^\ell K_\ell(x) = 0, \quad \text{and} \quad \lim_{x \rightarrow 0} x^\ell K_\ell(x) = 2^{\ell-1} \Gamma(\ell). \quad (8.26)$$

	$U_D(r)/(\mu(D)G_{\text{eff}}M)$	$\gamma_D$
$m_g = m_s = +\infty$	$-1/r^{D-3}$	$1/(D-3)$
$m_s = +\infty,  m_g r  \ll 1$	$+1/[(D-3)(D-1)r^{D-3}]$	$-1$
$m_s = 0, m_g = +\infty$	$-(D-2)^2/[(D-3)(D-1)r^{D-3}]$	$1/(D-2)$
$m \equiv m_g = m_s$	$- \left[ 1 - \frac{(D-3)\Omega_{D-2}}{(2\pi)^{(D-1)/2}} (mr)^{\frac{D-3}{2}} K_{\frac{D-3}{2}}(mr) \right] / r^{D-3}$	$1/(D-3)$

Table 4: Newton’s potential and  $\gamma(r)$  in higher dimensions  $D \geq 4$  for various values of the masses of the extra modes.

Note also that when one of the modes is absent, the divergence of  $U_D(r)$  at  $r = 0$  becomes stronger than in the generic case (8.23) — namely, of order  $1/r^{D-3}$  instead of  $1/r^{D-5}$ .

## 9 Gravitational waves

In this section we study the emission and propagation of gravitational radiation from sources in a general four-dimensional theory of the form (1.1) using the results of section 2. Our main result is a new formula for the power emitted by a source as a function of the quadrupole moment and the scalar radiation — see (9.40) below. This generalizes the Einstein gravity result to general  $\mathcal{L}(\text{Riemann})$  theories. We point out that a previous expression obtained for  $f(R)$  gravities in [72] is incorrect, and provide the corrected expression — which is a particular case of our general result.

### 9.1 Polarization of gravitational waves

In the de Donder gauge (2.47), the relevant components of the metric perturbation decomposed as in (2.51) satisfy equations (2.48), (2.52) and (2.53). In the vacuum, these reduce to

$$\Box \hat{h}_{\mu\nu} = 0, \quad (\Box - m_g^2)t_{\mu\nu} = 0, \quad (\Box - m_s^2)\phi = 0. \quad (9.1)$$

Using the tracelessness of  $t_{\mu\nu}$ , the gauge condition (2.47) and equations (9.1) along with (2.51), one can show that  $\partial^\mu t_{\mu\nu} = 0$ . The gauge redundancy has not been completely exploited, as we still have the freedom to make gauge transformations  $h_{\mu\nu} \rightarrow h_{\mu\nu} + 2\partial_{(\mu}\xi_{\nu)}$  where  $\xi_\mu$  satisfies  $\Box \xi_\mu = 0$ . This freedom can be used to impose four additional conditions on  $\hat{h}_{\mu\nu}$ . In particular, we can set  $\hat{h} = 0$  and  $\hat{h}_{ti} = 0$ , which is called the *traceless-transverse gauge* ( $TT$ ). Observe that we cannot impose similar conditions on  $t_{\mu\nu}$  because we can only make transformations with a harmonic gauge parameter  $\xi_\mu$ , but  $t_{\mu\nu}$  is not harmonic because it is massive. Hence, no degrees of freedom in  $t_{\mu\nu}$  can be removed

with such a gauge transformation and, as a consequence, the massive particles conserve all their polarizations.

Let us now look for plane-wave solutions of frequency  $\omega$ ,

$$\hat{h}_{\mu\nu}^{TT} = A_{\mu\nu} e^{-ik_\mu x^\mu}, \quad t_{\mu\nu} = B_{\mu\nu} e^{-ip_\mu x^\mu}, \quad \phi = c e^{-iq_\mu x^\mu}, \quad (9.2)$$

where  $k_\mu = (\omega, k_i)$ ,  $p_\mu = (\omega, p_i)$ ,  $q_\mu = (\omega, q_i)$ . Equations (9.1) produce the following dispersion relations

$$k^2 = \omega^2, \quad p^2 = \omega^2 - m_g^2, \quad q^2 = \omega^2 - m_s^2. \quad (9.3)$$

Note that for the massive modes to propagate, the frequency must be greater than the corresponding mass, *i.e.*,  $\omega^2 > m_g^2$  and  $\omega^2 > m_s^2$  respectively. Otherwise, the wave will be damped. Now, since we are working in the  $TT$  gauge, the polarization tensor  $A_{\mu\nu}$  satisfies the following constraints

$$A_{t\mu} = 0, \quad k^i A_{ij} = 0, \quad A_{ii} = 0, \quad (9.4)$$

which leave us with only two independent polarizations  $A_{\mu\nu}^+$  and  $A_{\mu\nu}^\times$ . On the other hand,  $B_{\mu\nu}$  only satisfies the constraints

$$p^\mu B_{\mu\nu} = 0, \quad \eta^{\mu\nu} B_{\mu\nu} = 0. \quad (9.5)$$

There are five degrees of freedom which correspond to the choice of a spatial part of the polarization,  $B_{ij}$ , satisfying

$$p^i p^j B_{ij} = \omega^2 B_{ii}, \quad (9.6)$$

which include the  $+$  and  $\times$  polarizations plus three additional ones. The time components are then given by

$$B_{tt} = B_{ii}, \quad B_{ti} = \frac{p_i}{\omega} B_{ij}. \quad (9.7)$$

Finally, from (2.51) it follows that the contribution to the metric perturbation associated to the scalar mode is given by  $\sim C_{\mu\nu} e^{-iq_\alpha x^\alpha}$  with polarization tensor

$$C_{\mu\nu} = \eta_{\mu\nu} - \frac{2q_\mu q_\nu}{m_s^2}, \quad (9.8)$$

which is linearly independent from  $A_{\mu\nu}$  and  $B_{\mu\nu}$  because it is not traceless.

In sum, gravitational waves in higher-order gravity can propagate up to six different polarizations — one for the scalar and five for the massive and massless gravitons. However, it is important to note that the massive modes do not propagate at lower frequencies, so the possible polarizations depend on the frequency.

## 9.2 Gravitational radiation from sources

Let us now consider a source  $T_{\mu\nu}(t, \vec{x})$  concentrated in a region whose diameter is much smaller than the distance  $r$  to the observer and which moves at a non-relativistic characteristic speed. Under such approximations

$$|\vec{x} - \vec{x}'| \approx r, \quad \frac{d\vec{x}}{dt} \ll 1, \quad (9.9)$$

the solutions in (8.2) can be further simplified. In particular, for the massless graviton  $\hat{h}_{\mu\nu}$  one finds

$$\hat{h}_{\mu\nu} = \frac{4G_{\text{eff}}}{r} \int d^3\vec{x}' T_{\mu\nu}(t-r, \vec{x}'). \quad (9.10)$$

Our interest here is in the radiative contributions of the solutions, *i.e.*, the ones which change with time. For gravitational waves, the time components  $\hat{h}_{\mu 0}$  are determined by the purely space-like ones, so we only need to compute those. The spatial components are radiative in general, and for them one finds the well-known quadrupole formula

$$\int d^3\vec{x}' T_{ij}(t-r, \vec{x}') = \frac{1}{2} \ddot{q}_{ij}(t-r), \quad (9.11)$$

where  $q_{ij}$  is the quadrupole moment of the source

$$q_{ij}(t-r) = \int d^3\vec{x} x^i x^j \rho(t-r, \vec{x}), \quad (9.12)$$

$\rho$  is the energy density and each dot denotes a time derivative. Therefore, the radiative part of  $\hat{h}_{\mu\nu}$  is given by

$$\hat{h}_{ij} = \frac{2G_{\text{eff}}}{r} \ddot{q}_{ij}(t-r). \quad (9.13)$$

Obviously, in the case of Einstein gravity — or for Einstein-like theories — this is the end of the story. However, in general  $\mathcal{L}(\text{Riemann})$  theories, we also have to take into account the additional modes. For the scalar  $\phi$  one finds

$$\phi = \frac{4G_{\text{eff}}}{r} \int d^3\vec{x}' T(t-r, \vec{x}') - 4G_{\text{eff}} m_s \int_r^\infty dt' \frac{J_1(m_s \sqrt{t'^2 - r^2})}{\sqrt{t'^2 - r^2}} \int d^3\vec{x}' T(t-t', \vec{x}'), \quad (9.14)$$

where  $J_1(x)$  is a Bessel function of the first kind. The integration of the trace yields

$$\int d^3\vec{x}' T(t-r, \vec{x}') = \int d^3\vec{x}' (-T_{00}(t-r, \vec{x}') + T_{ii}(t-r, \vec{x}')) = -M_0 - E_k(t-r) + \frac{1}{2} \ddot{q}_{ii}(t-r), \quad (9.15)$$

where  $M_0$  is the rest mass and  $E_k$  is the kinetic energy of the source. Since the rest mass is constant, it does not source any radiation, and the radiative part of the field is

$$\begin{aligned} \phi = & \frac{4G_{\text{eff}}}{r} \left( \frac{1}{2} \ddot{q}_{ii}(t-r) - E_k(t-r) \right) \\ & - 4G_{\text{eff}} m_s \int_r^\infty dt' \frac{J_1(m_s \sqrt{t'^2 - r^2})}{\sqrt{t'^2 - r^2}} \left( \frac{1}{2} \ddot{q}_{ii}(t-r) - E_k(t-r) \right). \end{aligned} \quad (9.16)$$

It is important to note that this field does not always radiate. Indeed, if one considers the source to be a set of point-like particles or a pressure-less perfect fluid (dust) then one gets  $\frac{1}{2} \ddot{q}_{ii}(t-r) - E_k(t-r) = \text{constant}$ <sup>25</sup>.

Finally, we have to determine the radiative part of  $t_{\mu\nu}$ . From (8.2) we can express this field as

$$t_{\mu\nu} = -H_{\langle\mu\nu\rangle} - \frac{1}{3m_g^2} \partial_{\langle\mu} \partial_{\nu\rangle} H, \quad (9.17)$$

where the purely spacelike components of  $H_{\mu\nu}$  for far sources are given by

$$H_{ij} = -\frac{2G_{\text{eff}}}{r} \ddot{q}_{ij}(t-r) + 2G_{\text{eff}} m_g \int_r^\infty dt' \frac{J_1(m_g \sqrt{t'^2 - r^2})}{\sqrt{t'^2 - r^2}} \ddot{q}_{ij}(t-t'). \quad (9.18)$$

Moreover, in the vacuum we get  $0 = \partial_\mu t^{\mu\nu} = \partial_\mu H^{\mu\nu}$ , so this allows us to characterize all the components of  $H_{\mu\nu}$  and  $t_{\mu\nu}$ <sup>26</sup>. By using (2.51), (9.17) and the solutions for  $\hat{h}_{ij}$ ,  $\phi$  and  $H_{ij}$  that we have just found, the full metric perturbation can be computed. Note that the perturbation at a distance  $r$  depends on the radiation emitted at all times previous to  $t-r$  and not only on the radiation emitted at the time  $t-r$ . This is related to the fact that the massive graviton and the scalar do not propagate at the speed of light. Indeed, according to the dispersion relation  $\omega = \sqrt{m_{g,s}^2 + k^2}$ , a wave packet with a central frequency  $\omega$  will travel at a velocity

$$v_{g,s} = \sqrt{1 - \frac{m_{g,s}^2}{\omega^2}}. \quad (9.19)$$

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<sup>25</sup>The energy-momentum tensor of a pressure-less fluid has the form  $T_{\mu\nu} = \rho u^\mu u^\nu$ , where  $\rho$  is the energy density and  $u^\mu$  is the 4-velocity field, satisfying  $u^\mu u_\mu = -1$ . Therefore  $T = -\rho$  and its integral yields the rest mass of the system. The same argument works for a set of point-like particles. Also, an explicit computation in that case shows that — at least — when particles interact only gravitationally, then  $\frac{1}{2} \ddot{q}_{ii} - E_k = E_k + E_p$ , where  $E_p$  is the gravitational potential energy of the system, and the previous quantity is a constant of motion.

<sup>26</sup>For example, for a plane wave solution we have  $p^\mu H_{\mu\nu} = 0$ , so we obtain the time-like components in terms of the purely space-like ones:  $H_{0i} = p^j H_{ij}/\omega$ ,  $H_{00} = p^i p^j H_{ij}/\omega^2$ . In the general case, the relations that we obtain are not algebraic but differential.

## Harmonic source

Let us work out explicitly the case corresponding to a source with harmonic motion. Then, the quadrupole moment takes the form  $q_{ij}(t) = a_{ij}e^{-i\omega t} + c_{ij}$ , where  $a_{ij}$  is the polarization tensor and  $c_{ij}$  is some plausible constant term. We also assume that the kinetic energy can be expressed as  $E_k = E_{k0}e^{-i\omega t}$ , plus a possible constant term which does not produce radiation and we neglect. For this kind of time-dependence, the integrals above can be computed and the fields take the following form

$$\hat{h}_{ij} = -\frac{2G_{\text{eff}}\omega^2}{r}e^{-i\omega(t-r)}a_{ij}, \quad (9.20)$$

$$H_{ij} = -\frac{2G_{\text{eff}}\omega^2}{r}e^{-i\omega t + i\sqrt{\omega^2 - m_g^2}r}a_{ij}, \quad (9.21)$$

$$\phi = -\frac{4G_{\text{eff}}\omega^2}{r}e^{-i\omega t + i\sqrt{\omega^2 - m_s^2}r}\left(\frac{1}{2}a_{ii} - E_{k0}\right). \quad (9.22)$$

Here, it is evident that the massive graviton and the scalar propagate only when  $\omega^2 > m_g^2$  or  $\omega^2 > m_s^2$  respectively. These expressions can be written in a more compact and suggestive way as

$$\hat{h}_{ij} = \frac{2G_{\text{eff}}}{r}\ddot{q}_{ij}(t-r), \quad H_{ij} = \frac{2G_{\text{eff}}}{r}\ddot{q}_{ij}(t-v_g r), \quad \phi = \frac{4G_{\text{eff}}}{r}\left(\frac{1}{2}\ddot{q}_{ii}(t-v_s r) - E_k(t-v_s r)\right), \quad (9.23)$$

where  $v_g$  and  $v_s$  are the group velocities of the massive graviton and the scalar, respectively<sup>27</sup> (9.19). Note that, while the expression for  $\hat{h}_{ij}$  is actually valid in general, the formulas for  $H_{ij}$  and  $\phi$  are only exact when the source is harmonic.

## 9.3 Power radiated by sources

In this subsection we derive the formula for the power emitted by some system in the form of gravitational radiation for a general theory of the form (1.1). In order to do so, we need to find the energy carried by gravitational waves. There are several ways of doing this. For instance, one can interpret the gravitational equations (2.1) with its linear part in  $h_{\mu\nu}$  subtracted — *i.e.*,  $\mathcal{E}_{\mu\nu} - \mathcal{E}_{\mu\nu}^L$  — as the gravitational stress-energy tensor, for which one needs to compute the equations of motion up to quadratic order [73]. We will use a different approach here. As we saw in section 2.4, it is possible to derive the linearized equations (2.27) from the quadratic action (2.55). From this, we can construct the canonical energy momentum tensor  $\tau_{\mu\nu}$  associated to  $h_{\mu\nu}$  using the

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<sup>27</sup>Note that for this kind of dispersion relation, the group velocity (which is the physical one) is the inverse of the phase velocity and that is why it seems that the velocity is in the wrong place.



Noether prescription — *e.g.*, [74]

$$\tau_{\mu\nu} = - \left[ \frac{\partial \mathcal{L}}{\partial(\partial^\mu h_{\alpha\beta})} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial^\mu \partial_\sigma h_{\alpha\beta})} \right] \partial_\nu h_{\alpha\beta} - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \partial_\sigma h_{\alpha\beta})} \partial_\sigma \partial_\nu h_{\alpha\beta} + \eta_{\mu\nu} \mathcal{L}. \quad (9.24)$$

By construction, the total energy-momentum conservation law holds

$$\partial_\mu (\tau^{\mu\nu} + T^{\mu\nu}) = 0. \quad (9.25)$$

Here  $T^{\mu\nu}$  is the stress tensor of matter (2.2), so  $\tau_{\mu\nu}$  can be used to determine the gravitational energy flux from a source<sup>28</sup>. This tensor can be computed explicitly, but we will not need its general expression here. Instead, we will make the further assumption that the perturbation modes are plane waves (9.2). In that case, if the perturbations  $\hat{h}_{\mu\nu}$ ,  $t_{\mu\nu}$  and  $\phi$  appeared separately in  $\mathcal{L}$ , the stress tensor for each of them would be given by

$$\tau_{\mu\nu}(\hat{h}_{\mu\nu}) = \frac{k_\mu k_\nu}{32\pi G_{\text{eff}}} \langle \hat{h}^{\alpha\beta} \hat{h}_{\alpha\beta} - \frac{1}{2} \hat{h}^2 \rangle, \quad (9.26)$$

$$\tau_{\mu\nu}(t_{\mu\nu}) = -\frac{1}{32\pi G_{\text{eff}}} p_\mu p_\nu \langle t^{\alpha\beta} t_{\alpha\beta} \rangle, \quad (9.27)$$

$$\tau_{\mu\nu}(\phi) = \frac{1}{192\pi G_{\text{eff}}} q_\mu q_\nu \langle \phi^2 \rangle, \quad (9.28)$$

where we have averaged the resulting expressions over space-time dimensions large compared with  $1/\omega$ <sup>29</sup>. This averaging<sup>30</sup> has the effect of killing crossed terms like<sup>31</sup>  $\hat{h}^{\alpha\beta} t_{\alpha\beta}$ ,  $\hat{h}\phi$ , which otherwise would be present in the final expression of  $\tau_{\mu\nu}$ . In that case, one simply finds  $\tau_{\mu\nu} = \tau_{\mu\nu}(\hat{h}) + \tau_{\mu\nu}(t) + \tau_{\mu\nu}(\phi)$ . Note that while  $\hat{h}_{\mu\nu}$  and  $\phi$  carry positive energy, the massive graviton  $t_{\mu\nu}$  propagates negative energy, which is in agreement with its ghost behavior. Now, the total radiated power crossing a sphere of radius  $r$  is given by

$$P = \int d\Omega r^2 \tau_{0i} n^i, \quad (9.29)$$

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<sup>28</sup>In the non-linear regime one can construct a gravitational energy-momentum pseudotensor by using the same prescription as in (9.24), namely:  $\tau_{\mu\nu}^{\text{non-linear}} = - \left[ \frac{\partial \mathcal{L}}{\partial(\partial^\mu g_{\alpha\beta})} - \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial^\mu \partial_\sigma g_{\alpha\beta})} \right] \partial_\nu g_{\alpha\beta} - \frac{\partial \mathcal{L}}{\partial(\partial^\mu \partial_\sigma g_{\alpha\beta})} \partial_\sigma \partial_\nu g_{\alpha\beta} + \eta_{\mu\nu} \mathcal{L}$ . Although this quantity is not a tensor, Noether's theorem ensures that  $\partial^\mu \left[ \sqrt{|g|} (\tau_{\mu\nu}^{\text{non-linear}} + T_{\mu\nu}) \right] = 0$  [75]. In the linear regime, these expressions reduce to (9.24) and (9.25) respectively.

<sup>29</sup>Hence, we are implicitly assuming  $r \gg 1/\omega$ .

<sup>30</sup>This averaging is the natural way of defining the energy and momentum of a wave, as it removes oscillations, see *e.g.*, [73, 76].

<sup>31</sup>As long as  $0 \neq m_s \neq m_g \neq 0$ .

where  $n^i$  is the unit vector normal to the sphere and note that with this definition a positive power means that the source loses energy. In order to perform the integration, we have to write the expressions above in terms of the spacelike components of the perturbations. In the case of  $\hat{h}_{\mu\nu}$  we can write  $\tau_{0i}$  for a harmonic wave as

$$\tau_{0i}(\hat{h}_{\mu\nu}) = \frac{n_i}{32\pi G_{\text{eff}}} \langle \dot{\hat{h}}^{\alpha\beta} \dot{\hat{h}}_{\alpha\beta} - \frac{1}{2} \dot{\hat{h}}^2 \rangle, \quad (9.30)$$

where we used the relation  $\omega^2 \langle \hat{h}^{\alpha\beta} \hat{h}_{\alpha\beta} \rangle = \langle \dot{\hat{h}}^{\alpha\beta} \dot{\hat{h}}_{\alpha\beta} \rangle$ . Now, since  $\hat{h}_{\mu\nu}$  is transverse,  $k^\mu \hat{h}_{\mu\nu} = 0$ , we can write  $\hat{h}_{00} = n^i n^j \hat{h}_{ij}$  and  $\hat{h}_{0i} = n^j \hat{h}_{ij}$ , so (9.30) takes the form

$$\tau_{0l}(\hat{h}_{\mu\nu}) = \frac{n_l}{32\pi G_{\text{eff}}} \left\langle \dot{\hat{h}}^{ij} \dot{\hat{h}}_{ij} - \frac{1}{2} \dot{\hat{h}}_{ii}^2 + n^i n^i (\dot{\hat{h}}_{ij} \dot{\hat{h}}_{kk} - 2 \dot{\hat{h}}_{ik} \dot{\hat{h}}_{jk}) + \frac{1}{2} (n^i n^j \dot{\hat{h}}_{ij})^2 \right\rangle. \quad (9.31)$$

Finally, using (9.13) and performing the integration over the solid angle yields the power radiated by the massless graviton  $h_{\mu\nu}$  in terms of the quadrupole moment

$$P(\hat{h}_{\mu\nu}) = \frac{G_{\text{eff}}}{5} \left\langle \ddot{q}^{ij} \ddot{q}_{ij} - \frac{1}{3} (\ddot{q}_{ii})^2 \right\rangle. \quad (9.32)$$

This is the well-known result found for Einstein gravity [73, 76], and the final answer for Einstein-like theories as defined in section 3.

For general theories, we need to compute the contributions from the extra modes, which we perform along the same lines. First, we note that it is convenient to write  $t_{\alpha\beta}$  in terms of the auxiliary field  $H_{\alpha\beta}$  (9.17), so we get *e.g.*,  $t_{\alpha\beta} t^{\alpha\beta} = H_{\alpha\beta} H^{\alpha\beta} - \frac{1}{3} H^2$ . Using this, we can write  $\tau_{0i}(t_{\mu\nu})$  as

$$\tau_{0i}(t_{\mu\nu}) = -\frac{n_i v_g}{32\pi G_{\text{eff}}} \left\langle \dot{H}^{\alpha\beta} \dot{H}_{\alpha\beta} - \frac{1}{3} \dot{H}^2 \right\rangle, \quad (9.33)$$

where we have taken into account that  $p_i = v_g \omega n_i$ , and again we have reabsorbed the  $\omega$  factor in a time derivative. Since  $H_{\alpha\beta}$  is also transverse,  $p^\alpha H_{\alpha\beta} = 0$ , we have  $H_{00} = v_g^2 n^i n^j H_{ij}$ ,  $H_{0i} = v_g n^j H_{ij}$ , and hence we find

$$\tau_{0l}(t_{\mu\nu}) = -\frac{n_l v_g}{32\pi G_{\text{eff}}} \left\langle \dot{H}^{ij} \dot{H}_{ij} - \frac{1}{3} \dot{H}_{ii}^2 + v_g^2 n^i n^j \left( \frac{2}{3} \dot{H}_{ij} \dot{H}_{kk} - 2 \dot{H}_{ik} \dot{H}_{jk} \right) + \frac{2}{3} v_g^4 (n^i n^j \dot{H}_{ij})^2 \right\rangle. \quad (9.34)$$

Now we can already perform the integral over the solid angle, and by using (9.23), we get

$$P(t_{\mu\nu}) = -\frac{G_{\text{eff}}}{5} \left\langle \left( \frac{5}{2} v_g - \frac{5}{3} v_g^3 + \frac{2}{9} v_g^5 \right) \ddot{q}^{ij} \ddot{q}_{ij} - \frac{1}{3} \left( \frac{5}{2} v_g - \frac{5}{3} v_g^3 - \frac{1}{3} v_g^5 \right) (\ddot{q}_{ii})^2 \right\rangle. \quad (9.35)$$

One can see that this flux is always negative, provided  $0 \leq v_g \leq 1$ . As a consequence, every time one of these modes is emitted, some positive energy must be added to the source in order to keep the total energy constant. In other words, the massive graviton would have the effect of making moving sources soak up gravitational radiation from the environment instead of emitting it! This is yet another manifestation of the ghost-nature of this mode.

Note also that this power does not cancel the one for the massless graviton, even if we set  $v_g = 1$  — corresponding to  $m_g = 0$ . There is no contradiction in this, since the polarization modes of  $t_{\mu\nu}$  are different from those of  $\hat{h}_{\mu\nu}$  and therefore the energy carried by these fields does not have to be necessarily opposite — and indeed, it is not. Note that this result is valid only when the perturbation propagates, *i.e.*, when  $\omega^2 > m_g^2$ . Otherwise, there is no emission of energy and  $P(t_{\mu\nu}) = 0$ . Thus, we can always use the previous formula with the convention  $v_g = 0$  if  $\omega^2 < m_g^2$ .

Finally, we can evaluate the power emitted by the scalar mode. The integral over the solid angle can be done straightforwardly, and the result is

$$P(\phi) = \frac{G_{\text{eff}} v_s}{3} \left\langle \left( \frac{1}{2} \ddot{q}_{ii} - \dot{E}_k \right)^2 \right\rangle. \quad (9.36)$$

As stated previously, the scalar radiation vanishes as long as we consider our system to be composed of dust, or non-interacting particles (without interactions different from gravity). For example, a binary — see the next epigraph — is very approximately a system of this kind, so there is no scalar radiation in that case. The scalar radiation only plays a role in systems where other interactions different from gravity are important, like in the explosion of a supernova [77]. Now, the final result for the power emitted in the form of gravitational waves in a theory of the form (1.1) reads

$$P = \frac{G_{\text{eff}}}{5} \left\langle \left( 1 - \frac{5}{2} v_g + \frac{5}{3} v_g^3 - \frac{2}{9} v_g^5 \right) \ddot{q}^{ij} \ddot{q}_{ij} - \frac{1}{3} \left( 1 - \frac{5}{2} v_g + \frac{5}{3} v_g^3 + \frac{1}{3} v_g^5 \right) (\ddot{q}_{ii})^2 + \frac{5}{3} v_s \left( \frac{1}{2} \ddot{q}_{ii} - \dot{E}_k \right)^2 \right\rangle, \quad (9.37)$$

where

$$v_{g,s} = \begin{cases} \sqrt{1 - \frac{m_{g,s}^2}{\omega^2}} & \text{if } \omega^2 \geq m_{g,s}^2, \\ 0 & \text{if } \omega^2 < m_{g,s}^2. \end{cases} \quad (9.38)$$

If we decompose the quadrupole moment into its trace and traceless part,

$$q_{ij} = Q_{ij} + \frac{1}{3} \delta_{ij} q_{kk}, \quad (9.39)$$

we can rewrite this expression as

$$P = \frac{G_{\text{eff}}}{5} \left\langle \left( 1 - \frac{5}{2}v_g + \frac{5}{3}v_g^3 - \frac{2}{9}v_g^5 \right) \ddot{Q}^{ij}\ddot{Q}_{ij} - \frac{5}{27}v_g^5(\ddot{q}_{ii})^2 + \frac{5}{3}v_s \left( \frac{1}{2}\ddot{q}_{ii} - \dot{E}_k \right)^2 \right\rangle. \quad (9.40)$$

Note that in Einstein gravity, the result only involves the traceless part of  $q_{ij}$  [73, 76], while here we also have contributions from its trace and from the variation of the source kinetic energy due to the presence of extra modes. Let us stress again that (9.40) is valid only for a harmonic source. In the case of a more general time dependence,  $q_{ij}$  and  $E_k$  can be Fourier-expanded and then the power of each Fourier mode can be extracted from (9.40). The total power would then be the sum of all of those contributions.

Equation (9.40) is the main result of this section. It generalizes the Einstein gravity formula (9.32) to general  $\mathcal{L}(\text{Riemann})$  theories. A previous extension of (9.32) to  $f(R)$  gravity was found in [72]. For  $f(R)$ , our formula above reduces to

$$P_{f(R)} = \frac{G}{5f'(\bar{R})} \left\langle \ddot{Q}^{ij}\ddot{Q}_{ij} + \frac{5}{3}v_s \left( \frac{1}{2}\ddot{q}_{ii} - \dot{E}_k \right)^2 \right\rangle, \quad (9.41)$$

where  $v_s$  reads — see appendix C,

$$v_s = \begin{cases} \sqrt{1 - \frac{m_s^2}{\omega^2}} & \text{if } \omega^2 \geq m_s^2, \\ 0 & \text{if } \omega^2 < m_s^2, \end{cases} \quad \text{where} \quad m_s^2 = \frac{f'(\bar{R}) - \bar{R}f''(\bar{R})}{3f''(\bar{R})}. \quad (9.42)$$

This expression disagrees with the one found in [72] — see (82) in that paper. However, it is easy to see that the second term on the RHS of equation (43) in [72] is identically zero, so the second contribution on the RHS of (82) is absent. Similarly, the first term in their (82) is missing an overall<sup>32</sup>  $1/f'(\bar{R})^2$ . And finally, the authors seem to have ignored the contribution from the scalar mode, which explains why they do not find the term proportional to  $(1/2\ddot{q}_{ii} - \dot{E}_k)^2$ .

## Binary system

As an application of (9.40), let us compute explicitly the power radiated by a system consisting of two masses  $m_1$  and  $m_2$  separated by a distance  $r$  in a circular orbit contained in the plane  $z = 0$ . For this kind of system, the position of the masses is given by

$$\vec{x}_1(t) = \frac{rm_2}{m_1 + m_2} (\cos(\Omega t), \sin(\Omega t), 0), \quad (9.43)$$

$$\vec{x}_2(t) = -\frac{rm_1}{m_1 + m_2} (\cos(\Omega t), \sin(\Omega t), 0), \quad (9.44)$$

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<sup>32</sup>This seems to arise from a wrong identification in (48). Note that equations (46)-(50) in [72] are also inconsistent with each other.

where the orbital frequency  $\Omega$  reads

$$\Omega^2 = \frac{G_{\text{eff}}(m_1 + m_2)}{r^3}. \quad (9.45)$$

Assuming the masses to be point-like, the mass density can be written as  $\rho(\vec{x}, t) = m_1\delta(\vec{x} - \vec{x}_1(t)) + m_2\delta(\vec{x} - \vec{x}_2(t))$ . Then, the quadrupole moment (9.12) is

$$q_{ij}(t) = \frac{r^2 m_1 m_2}{2(m_1 + m_2)} \begin{pmatrix} 1 + \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1 - \cos(2\Omega t) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (9.46)$$

The trace and the kinetic energy are constant,  $q_{ii} = (r^2 m_1 m_2)/(m_1 + m_2)$ ,  $\dot{E}_k = 0$ , so there is no scalar radiation in this case<sup>33</sup>. The traceless part of  $q_{ij}$  reads in turn

$$Q_{ij}(t) = \frac{r^2 m_1 m_2}{2(m_1 + m_2)} \begin{pmatrix} 1/3 + \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ \sin(2\Omega t) & 1/3 - \cos(2\Omega t) & 0 \\ 0 & 0 & -2/3 \end{pmatrix}. \quad (9.47)$$

Applying (9.40) we obtain the following result

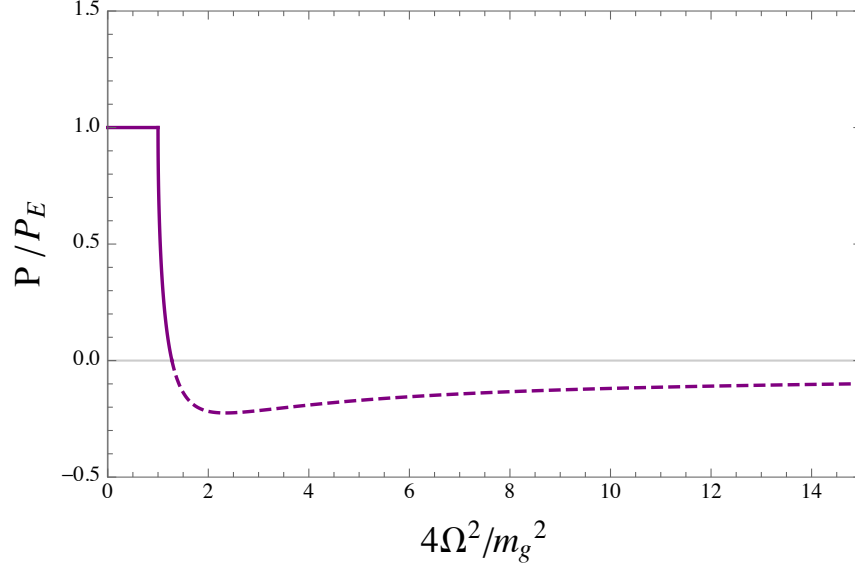


Figure 3: Power emitted by a binary system for a theory of the form (1.1),  $P/P_E$  as a function of  $4\Omega^2/m_g^2$ .

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<sup>33</sup>In the case of a more general orbit there is no scalar radiation either, because, as discussed earlier,  $\frac{1}{2}\ddot{q}_{ii} - E_k = E_k + E_p = \text{constant}$  for that kind of system.

$$P = P_E \left( 1 - \sqrt{1 - \frac{m_g^2}{4\Omega^2} \left[ \frac{19}{18} + \frac{11}{36} \frac{m_g^2}{\Omega^2} + \frac{1}{72} \frac{m_g^4}{\Omega^4} \right]} \right), \quad (9.48)$$

where

$$P_E = \frac{32G_{\text{eff}}^4 m_1^2 m_2^2 (m_1 + m_2)}{5r^5}, \quad (9.49)$$

is the result corresponding to theories which do not propagate the massive graviton — see section 3. In particular, (9.49) is the Einstein gravity result when  $G_{\text{eff}} = G$ . Expression (9.48) is valid for  $4\Omega^2 > m_g^2$ . When  $4\Omega^2 < m_g^2$  instead, the result reduces to  $P_E$  — see Fig. 3. When  $4\Omega^2 = m_g^2$ , the effect of the massive graviton makes the power start decreasing. In particular, when  $4\Omega^2/m_g^2 \simeq 1.2761$ , the power emitted vanishes. For even smaller values of  $m_g^2$  with respect to  $\Omega^2$ , the power becomes negative acquiring its minimum value at  $4\Omega^2/m_g^2 = 1 + 3/\sqrt{5}$ , for which  $P/P_E(1 + 3/\sqrt{5}) = 1 - \sqrt{3/2} \simeq -0.2247$ . Finally, for  $\Omega^2 \gg m_g^2$ , the power tends to the constant value  $P/P_E(\Omega^2 \gg m_g^2) = -1/18 \simeq -0.0556$ . Given a theory with  $m_g^2 < \infty$ , there would exist a critical frequency  $\Omega_c^2 \simeq 0.31903m_g^2$  for which the source would stop emitting radiation and such that for greater frequencies the source would start absorbing radiation instead of emitting it. This exotic process should not be regarded as physical and illustrates the pathological character of the class of theories which propagate the additional spin-2 mode.

## 10 Wald formalism for general $\mathcal{L}(\text{Riemann})$ theories

In this section we present a self-contained review of Wald’s formalism [40] applied to general  $\mathcal{L}(\text{Riemann})$  theories. Wald’s formalism provides a systematic way of constructing conserved quantities in diffeomorphism invariant theories. It was originally developed to derive the first law of black hole mechanics for generic theories of gravity [41, 42], but it has led to many interesting applications, *e.g.*, in holography [47, 78, 79]. Our discussion is mainly based on [41, 80], where this formalism was developed for higher-derivative theories of gravity. Here we present new results for the symplectic structure  $\omega$  and the surface charge  $\delta Q_\xi - \xi \cdot \Theta$  for  $\mathcal{L}(\text{Riemann})$  theories. Throughout this section we set  $L_{\text{matter}} = 0$  in (1.1), *i.e.*, we assume that the Lagrangian does not depend on any matter fields. In appendix E we provide explicit expressions for the quantities considered in this section for some relevant theories.

### Lagrangian and symplectic potential

The starting point of the Wald formalism is a diffeomorphism covariant Lagrangian, which — in our case — is assumed to be a local functional of the metric and the Riemann tensor. The Lagrangian is treated as a  $D$ -form on the  $D$ -dimensional spacetime manifold

$\mathcal{M}$ , namely

$$\mathbf{L} = \mathcal{L}(R_{\mu\nu\rho\sigma}, g^{\alpha\beta})\epsilon, \quad (10.1)$$

where  $\mathcal{L}(R_{\mu\nu\rho\sigma}, g^{\alpha\beta})$  is the Lagrangian density and  $\epsilon$  is the volume form on  $\mathcal{M}$ . For future reference, we will be using the following shorthand notation for the volume form of any codimension- $n$  submanifold

$$\epsilon_{\mu_1 \dots \mu_n} \equiv \frac{1}{(D-n)!} \sqrt{-g} \epsilon_{\mu_1 \dots \mu_n \nu_{n+1} \dots \nu_D} dx^{\nu_{n+1}} \wedge \dots \wedge dx^{\nu_D}. \quad (10.2)$$

Under a variation of the metric<sup>34</sup>, the first-order variation of the Lagrangian is given by

$$\delta \mathbf{L} = \epsilon \mathcal{E}^{\mu\nu} \delta g_{\mu\nu} + d\Theta(g, \delta g), \quad (10.3)$$

where  $\mathcal{E}^{\mu\nu} = 0$  are the equations of motion for the theory, given by<sup>35</sup> (2.1), and  $\Theta$  is the boundary term that arises due to partial integration of terms involving derivatives of  $\delta g$ . The  $(D-1)$ -form  $\Theta$  is locally constructed from  $g$  and  $\delta g$ , and is called the *symplectic potential form*. From (10.3) it is clear that  $\Theta$  is not uniquely defined, since one always has the freedom to add a closed — and hence locally exact [81] — form to it. However, as shown in [41, 82], it is always possible to construct an explicit covariant formula for  $\Theta$  which fixes this ambiguity. For  $\mathcal{L}(\text{Riemann})$  theories this somewhat canonical formula reads [41, 80]

$$\Theta = \epsilon_\mu \left( 2P^{\mu\alpha\beta\nu} \nabla_\nu \delta g_{\alpha\beta} - 2\nabla_\nu P^{\mu\alpha\beta\nu} \delta g_{\alpha\beta} \right), \quad (10.4)$$

where  $P^{\mu\beta\alpha\nu}$  is defined in (2.2). Furthermore, by employing the relation

$$\nabla_\nu \delta g_{\alpha\beta} = g_{\beta\rho} \delta \Gamma_{\nu\alpha}^\rho + g_{\alpha\rho} \delta \Gamma_{\nu\beta}^\rho, \quad (10.5)$$

the symplectic potential form can also be written as

$$\Theta = \epsilon_\mu \left( -2P^{\mu\alpha\beta}{}_\nu \delta \Gamma_{\alpha\beta}^\nu - 2\nabla_\nu P^{\mu\alpha\beta\nu} \delta g_{\alpha\beta} \right), \quad (10.6)$$

where we used that  $P^{\mu\alpha\beta\nu}$  is antisymmetric in its last two indices,  $P^{\mu\alpha\beta\nu} = -P^{\mu\alpha\nu\beta}$ , which implies that  $P^{\mu\alpha\beta\nu} \delta \Gamma_{\beta\nu}^\rho = 0$ .

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<sup>34</sup>For convenience, we vary the Lagrangian with respect to the metric  $g_{\mu\nu}$ , although it was initially defined in terms of the inverse metric  $g^{\mu\nu}$ .

<sup>35</sup>Notice that the equations of motion with indices up and the one with indices down are related by a minus sign:  $\mathcal{E}^{\mu\nu} = -g^{\mu\alpha} g^{\mu\nu} \mathcal{E}_{\alpha\beta}$ .

## Symplectic form

The *symplectic current form* is defined as the antisymmetrized variation of  $\Theta$  [82]

$$\omega(g, \delta_1 g, \delta_2 g) \equiv \delta_1 \Theta(g, \delta_2 g) - \delta_2 \Theta(g, \delta_1 g). \quad (10.7)$$

From (10.3) and (10.7) it follows that  $\omega$  obeys the relation

$$d\omega = -\delta_1(\epsilon \mathcal{E}^{\mu\nu}) \delta_2 g_{\mu\nu} + \delta_2(\epsilon \mathcal{E}^{\mu\nu}) \delta_1 g_{\mu\nu}. \quad (10.8)$$

Here it was used that the exterior derivative  $d$  commutes with the variation  $\delta$ :  $d(\delta\Theta) = \delta(d\Theta)$ . Therefore, if  $\delta g$  satisfies the linearized equations of motion  $\delta(\epsilon \mathcal{E}^{\mu\nu}) = 0$ , then the symplectic current form is closed

$$d\omega = 0. \quad (10.9)$$

This relation implies — by Stokes's theorem — that the integral of  $\omega$  over a compact Cauchy surface  $\mathcal{C}$  is independent of the choice of  $\mathcal{C}$ . For non-compact Cauchy surfaces one has to impose appropriate boundary conditions on the metric and its perturbations on  $\partial\mathcal{C}$  in order to assure convergence of the integral. Here we just assume that such boundary conditions exist, so that the integral of  $\omega$  over a Cauchy surface  $\mathcal{C}$  is a conserved quantity. This quantity is called the *symplectic two-form* [41, 82]

$$\Omega(g, \delta_1 g, \delta_2 g) \equiv \int_{\mathcal{C}} \omega(g, \delta_1 g, \delta_2 g). \quad (10.10)$$

Let us explain the origin of its name. In fact,  $\Omega$  can be regarded as a two-form defined on the space of metric configurations  $\mathcal{F}$ . This is because  $\Omega$  is a local functional of the linearized perturbations  $\delta_1 g$  and  $\delta_2 g$ , where the variation  $\delta$  can be viewed as the exterior derivative on this space. Moreover, from (10.7) it follows that  $\Omega$  is closed, *i.e.*,  $\delta\Omega = 0$ , due to the fact that the exterior derivative satisfies the relation  $\delta^2 g = 0$ . Now a proper symplectic form on phase space is both closed and non-degenerate. The form (10.10) is degenerate — and is hence sometimes called the *presymplectic form* instead — but one can construct a non-degenerate two-form from (10.10) by modding out  $\mathcal{F}$  by the degeneracy subspace of  $\Omega$ . Then the non-degenerate  $\Omega$  and the solution submanifold of  $\mathcal{F}$  constitute a well-defined *covariant phase space* [82].

Let us now compute the symplectic current form explicitly for  $\mathcal{L}(\text{Riemann})$  theories. If we write the symplectic potential form as  $\Theta = \epsilon_\mu \Theta^\mu$ , then the definition of  $\omega$  (10.7) becomes

$$\omega(g, \delta_1 g, \delta_2 g) = \epsilon_\mu \left[ (\delta_1 \Theta^\mu(g, \delta_2 g)) + \frac{1}{2} g^{\mu\nu} \delta_1 g_{\mu\nu} \Theta^\mu(g, \delta_2 g) \right] - [1 \leftrightarrow 2], \quad (10.11)$$



where we used  $\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}$ . Next, one can insert two expressions for  $\Theta$  for  $\mathcal{L}(\text{Riemann})$  theories.  $\omega$  simplifies immediately if one inserts the second expression (10.6), since in that case one can employ the relation  $\delta_{[1}\delta_{2]}\Gamma_{\alpha\beta}^\nu = 0$ . If one inserts the first expression (10.4) instead, one has to be careful with evaluating the term  $4P^{\mu\alpha\beta\nu}\delta_{[1}(\nabla_\nu\delta_{2]}g_{\alpha\beta})$ , because the variation and the covariant derivative do not commute. In the latter case one can use the fact that the variation and the partial derivative commute *i.e.*,  $[\delta, \partial_a]f = 0$ . We checked that both procedures give the same answer. The result is

$$\begin{aligned} \omega = \epsilon_\mu \Big[ & - \left( 2\delta_1 P^{\mu\alpha\beta}{}_\nu + P^{\mu\alpha\beta}{}_\nu g^{\rho\sigma} \delta_1 g_{\rho\sigma} \right) \delta_2 \Gamma_{\alpha\beta}^\nu \\ & - \left( 2\delta_1 \nabla_\nu P^{\mu\alpha\beta\nu} + g^{\rho\sigma} \delta_1 g_{\rho\sigma} \nabla_\nu P^{\mu\alpha\beta\nu} \right) \delta_2 g_{\alpha\beta} \Big] - [1 \leftrightarrow 2]. \end{aligned} \quad (10.12)$$

By employing the formula for the variation of the Christoffel connection

$$\delta\Gamma_{\alpha\beta}^\nu = \frac{1}{2}g^{\mu\nu} (\nabla_\alpha \delta g_{\beta\mu} + \nabla_\beta \delta g_{\alpha\mu} - \nabla_\mu \delta g_{\alpha\beta}), \quad (10.13)$$

the result (10.12) can also be written as

$$\begin{aligned} \omega = \epsilon_\mu \Big[ & \left( 2\delta_1 P^{\mu\alpha\beta\nu} + (P^{\mu\nu\rho\beta} g^{\alpha\sigma} + P^{\mu\alpha\rho\nu} g^{\beta\sigma} + P^{\mu\alpha\beta\rho} g^{\nu\sigma} + P^{\mu\alpha\beta\nu} g^{\rho\sigma}) \delta_1 g_{\rho\sigma} \right) \nabla_\nu \delta_2 g_{\alpha\beta} \\ & - \left( 2\delta_1 \nabla_\nu P^{\mu\alpha\beta\nu} + g^{\rho\sigma} \delta_1 g_{\rho\sigma} \nabla_\nu P^{\mu\alpha\beta\nu} \right) \delta_2 g_{\alpha\beta} \Big] - [1 \leftrightarrow 2]. \end{aligned} \quad (10.14)$$

Finally, by inserting a formula for the variation of  $P^{\mu\alpha\beta\nu}$  that follows from (2.8),

$$\delta P^{\mu\alpha\beta\nu} = 2g^{\sigma[\mu} P^{\alpha]\rho\beta\nu} \delta g_{\rho\sigma} + g^{\lambda\gamma} g^{\eta\delta} C_{\rho\sigma\lambda\eta}^{\mu\alpha\beta\nu} \delta R^{\rho\sigma}{}_{\gamma\delta}, \quad (10.15)$$

where  $C_{\rho\sigma\lambda\eta}^{\mu\alpha\beta\nu}$  is defined by (2.4), the symplectic current form can be written as

$$\begin{aligned} \omega = \epsilon_\mu \Big[ & \left( S^{\mu\alpha\beta\nu\rho\sigma} \delta_1 g_{\rho\sigma} + 2g^{\lambda\gamma} g^{\eta\delta} C_{\rho\sigma\lambda\eta}^{\mu\alpha\beta\nu} \delta_1 R^{\rho\sigma}{}_{\gamma\delta} \right) \nabla_\nu \delta_2 g_{\alpha\beta} \\ & - \left( 2\delta_1 \nabla_\nu P^{\mu\alpha\beta\nu} + g^{\rho\sigma} \delta_1 g_{\rho\sigma} \nabla_\nu P^{\mu\alpha\beta\nu} \right) \delta_2 g_{\alpha\beta} \Big] - [1 \leftrightarrow 2], \end{aligned} \quad (10.16)$$

$$\begin{aligned} \text{with } S^{\mu\alpha\beta\nu\rho\sigma} \equiv & -2P^{\nu(\alpha\beta)(\rho} g^{\sigma)\mu} + 2P^{\mu\nu(\rho|(\alpha} g^{\beta)|\sigma)} \\ & + P^{\mu(\rho|\nu(\alpha} g^{\beta)|\sigma)} + P^{\mu(\alpha\beta)(\rho} g^{\sigma)\nu} + P^{\mu(\alpha\beta)\nu} g^{\rho\sigma}. \end{aligned} \quad (10.17)$$

To arrive at the expression for  $S^{\mu\alpha\beta\nu\rho\sigma}$  we employed the first Bianchi identity for  $P^{\mu\alpha\beta\nu}$ :  $P^{\mu\alpha\beta\nu} + P^{\mu\beta\nu\alpha} + P^{\mu\nu\alpha\beta} = 0$ . This new formula for the symplectic current form applies to any higher curvature gravity theory. Expressions for  $\omega$  were previously obtained for Einstein gravity [83–85] and  $f(R)$  gravity [86]. It can be checked that this formula provides the same results in those cases, as we show in appendix E.

## Noether current and Noether charge

Next let  $\xi$  be an arbitrary vector field on  $\mathcal{M}$  which generates an infinitesimal diffeomorphism. Since the Lagrangian (10.1) is diffeomorphism invariant, it varies under a diffeomorphism as

$$\delta_\xi \mathbf{L} = \mathcal{L}_\xi \mathbf{L} = d(\xi \cdot \mathbf{L}), \quad (10.18)$$

where in the last equality Cartan's magic formula was used:  $\mathcal{L}_\xi \mathbf{L} = \xi \cdot d\mathbf{L} + d(\xi \cdot \mathbf{L})$ . The first term vanishes since  $\mathbf{L}$  is a top form, and the dot in the second term denotes the interior product of the vector  $\xi$  with the form  $\mathbf{L}$ .

Since diffeomorphisms are local symmetries of the theory, one can associate a *Noether current* — represented as a  $(D-1)$ -form — to each vector field  $\xi$  [40, 82]

$$\mathbf{J}_\xi \equiv \Theta(g, \mathcal{L}_\xi g) - \xi \cdot \mathbf{L}. \quad (10.19)$$

It follows from (10.3) and (10.18) that the exterior derivative of  $\mathbf{J}_\xi$  is

$$d\mathbf{J}_\xi = -\epsilon \mathcal{E}^{\mu\nu} \mathcal{L}_\xi g_{\mu\nu}. \quad (10.20)$$

As a consequence, the Noether current form is closed if the equations of motion  $\mathcal{E}^{\mu\nu} = 0$  are satisfied. In that case, Poincaré's lemma implies that it is locally exact [81]. What is more, in the appendix of [87] it was shown that off shell  $\mathbf{J}_\xi$  can always be written in the form

$$\mathbf{J}_\xi = d\mathbf{Q}_\xi + \xi^\nu \mathbf{C}_\nu, \quad (10.21)$$

where  $\mathbf{Q}_\xi$  is called the *Noether charge*  $(D-2)$ -form and  $\mathbf{C}_\nu = 0$  are the constraint equations of the theory. For theories that only depend on the metric field, these equations are given by  $\mathbf{C}_\nu = 2\epsilon_\mu \mathcal{E}^\mu{}_\nu$  with  $\mathcal{E}^\mu{}_\nu \equiv g^{\mu\alpha} \mathcal{E}_{\alpha\nu}$ .

Although  $\mathbf{Q}_\xi$  is not uniquely determined by equation (10.21), there exists an explicit algorithm by [81] to construct  $\mathbf{Q}_\xi$  from  $\mathbf{J}_\xi$ . For  $\mathcal{L}$ (Riemann) theories of gravity this construction yields [41, 80]

$$\mathbf{Q}_\xi = \epsilon_{\mu\nu} (-P^{\mu\nu\rho\sigma} \nabla_\rho \xi_\sigma - 2\xi_\rho \nabla_\sigma P^{\mu\nu\rho\sigma}). \quad (10.22)$$

Thus, by equation (10.21) the Noether current form is

$$\mathbf{J}_\xi = \epsilon_\mu \left[ -2\nabla_\nu (P^{\mu\nu\rho\sigma} \nabla_\rho \xi_\sigma) - 4\nabla_\nu (\xi_\rho \nabla_\sigma P^{\mu\nu\rho\sigma}) + 2\mathcal{E}^\mu{}_\nu \xi^\nu \right]. \quad (10.23)$$

## Surface charge

From (10.3), (10.7), (10.19), and (10.21), one can obtain a fundamental identity

$$\omega(g, \delta g, \mathcal{L}_\xi g) = d\mathbf{k}_\xi(g, \delta g) + 2\delta(\epsilon_\mu \mathcal{E}^\mu{}_\nu) \xi^\nu + \xi^\lambda \epsilon_\lambda \mathcal{E}^{\mu\nu} \delta g_{\mu\nu}, \quad (10.24)$$

$$\text{where } \mathbf{k}_\xi(g, \delta g) \equiv \delta^{[g]} \mathbf{Q}_\xi(g) - \xi \cdot \Theta(g, \delta g) \quad (10.25)$$

is known as the *Iyer-Wald surface charge*  $(D-2)$ -form. Notice that this relation applies to arbitrary metrics  $g$ , metric perturbations  $\delta g$  and vector fields  $\xi$ . This identity was first established off shell by Wald [40], and for field dependent vector fields — *e.g.*, vector fields that depend on the metric  $\xi = \xi(g)$  — a proof can be found in the appendix of [78]. The variation  $\delta^{[g]}\mathbf{Q}_\xi \equiv \delta\mathbf{Q}_\xi - \mathbf{Q}_{\delta\xi}$  acts only on the explicit dependence on the metric and its derivatives in  $\mathbf{Q}_\xi$ , and not on the implicit dependence on  $\xi$ .

A special case of the identity occurs when  $\xi$  is an exact Killing vector. In that case the relation gives rise to the first law of black hole mechanics [40, 41]. Since  $\mathcal{L}_\xi g = 0$ , the left hand side of (10.24) vanishes, and if  $g$  and  $\delta g$  satisfy, respectively, the full equations of motion and the linearized ones, one obtains

$$d\mathbf{k}_\xi = 0. \quad (10.26)$$

Therefore, the integral of  $\mathbf{k}_\xi$  over a  $(D-2)$ -dimensional, spacelike compact surface  $S$  is “conserved”, in the sense that it is independent of the choice of  $S$ . If the normal directions to  $S$  are the time and radial direction, then the integral is the same at every time and radial coordinate. In order for this integral to be the variation of a finite conserved charge, certain integrability conditions should be satisfied [82].

Let us now compute this quantity for general  $\mathcal{L}(\text{Riemann})$  theories. Inserting the known expressions for  $\mathbf{Q}_\xi$  (10.22) and  $\Theta$  (10.4) into the definition of  $\mathbf{k}_\xi$  (10.25) yields

$$\begin{aligned} \mathbf{k}_\xi = \delta^{[g]} [\epsilon_{\mu\nu} (-P^{\mu\nu\rho} \nabla_\rho \xi^\sigma - 2\xi_\rho \nabla_\sigma P^{\mu\nu\rho\sigma})] \\ - \xi^\lambda \epsilon_{\mu\lambda} (2P^{\mu\alpha\beta\nu} \nabla_\nu \delta g_{\alpha\beta} - 2\nabla_\nu P^{\mu\alpha\beta\nu} \delta g_{\alpha\beta}) . \end{aligned} \quad (10.27)$$

By letting the variation act only on the explicit dependence on the metric, and collecting similar terms, we arrive at

$$\begin{aligned} \mathbf{k}_\xi = \epsilon_{\mu\nu} \Big[ -\delta P^{\mu\nu\rho} \nabla_\rho \xi^\sigma - 2\xi^\rho \delta(\nabla_\sigma P^{\mu\nu\rho\sigma}) \\ + \left( -\frac{1}{2} P^{\mu\nu\rho\sigma} g^{\alpha\beta} \nabla_\rho \xi_\sigma + 2\xi^\nu \nabla_\lambda P^{\mu\alpha\beta\lambda} - \xi_\rho \nabla_\sigma P^{\mu\nu\rho\sigma} g^{\alpha\beta} \right) \delta g_{\alpha\beta} \\ - (\xi^\alpha P^{\mu\nu\lambda\beta} + 2\xi^\nu P^{\mu\alpha\beta\lambda}) \nabla_\lambda \delta g_{\alpha\beta} \Big] . \end{aligned} \quad (10.28)$$

Here we have defined the  $\delta^{[g]}$  variation of the vector  $\xi^\sigma$  (with index up) to be zero, *i.e.*,  $\delta^{[g]}\xi^\sigma \equiv 0$ , which implies that  $\delta^{[g]}\xi_\sigma = \xi^\alpha \delta g_{\alpha\sigma}$  and  $\delta^{[g]}(\nabla_\rho \xi^\sigma) = \xi^\alpha \delta \Gamma_{\alpha\rho}^\sigma$ . Finally, introducing the variation of  $P^{\mu\nu\rho\sigma}$  (10.15) we obtain the expression

$$\begin{aligned} \mathbf{k}_\xi = \epsilon_{\mu\nu} \Big[ -g^{\lambda\gamma} g^{\delta\eta} C_{\alpha\beta\gamma\delta}^{\mu\nu\rho\sigma} \nabla_\rho \xi_\sigma \delta R^{\alpha\beta}{}_{\lambda\eta} - 2\xi^\rho \delta(\nabla_\sigma P^{\mu\nu\rho\sigma}) \\ + (P^{\mu\nu\alpha\lambda} \nabla^\beta \xi_\lambda - \frac{1}{2} P^{\mu\nu\rho\sigma} g^{\alpha\beta} \nabla_\rho \xi_\sigma + 2\xi^\nu \nabla_\lambda P^{\mu\alpha\beta\lambda} - \xi_\rho \nabla_\sigma P^{\mu\nu\rho\sigma} g^{\alpha\beta}) \delta g_{\alpha\beta} \\ - (\xi^\alpha P^{\mu\nu\lambda\beta} + 2\xi^\nu P^{\mu\alpha\beta\lambda}) \nabla_\lambda \delta g_{\alpha\beta} \Big] . \end{aligned} \quad (10.29)$$

## Barnich-Brandt-Compère definitions of $\omega$ and $\mathbf{k}_\xi$

A different method for constructing a covariant phase space was developed by Barnich, Brandt and Compère in [88–90]. Their definitions of the relevant quantities are based on the equations of motion rather than the Lagrangian. Hence their method is also universal, in the sense that it applies to any diffeomorphism invariant theory — in fact, their formalism is more general, since it holds for any theory with local gauge symmetries. Moreover, their definitions do not suffer from any ambiguities, as is the case for the Wald formalism — see the next epigraph. Most quantities agree with those defined by Lee, Wald and Iyer, except for the symplectic current  $\omega$  and the surface charge  $\mathbf{k}_\xi$ . For completeness, let us present here the Barnich-Brandt-Compère definitions of  $\omega$  and  $\mathbf{k}_\xi$  for  $\mathcal{L}(\text{Riemann})$  theories. A pedagogical review of this method can be found in [78, 80].

Firstly, the *Barnich-Compère symplectic current* — also known as *invariant* symplectic current — differs from the Lee-Wald definition (10.7) by an exact form

$$\omega^{\text{BC}}(g, \delta_1 g, \delta_2 g) = \omega^{\text{LW}}(g, \delta_1 g, \delta_2 g) - d\mathbf{E}(g, \delta_1 g, \delta_2 g), \quad (10.30)$$

where  $\mathbf{E}$  was computed for arbitrary higher derivative Lagrangians by [80]. We provide two equivalent expressions for  $\mathbf{E}$

$$\begin{aligned} \mathbf{E}(g, \delta_1 g, \delta_2 g) &= \epsilon_{\mu\nu} \frac{1}{2} \left[ -\frac{3}{2} P^{\mu\nu\rho\alpha} g^{\sigma\beta} + 2P^{\mu\rho\sigma\alpha} g^{\nu\beta} \right] \delta_1 g_{\rho\sigma} \delta_2 g_{\alpha\beta} - [1 \leftrightarrow 2] \\ &= \epsilon_{\mu\nu} \left[ -\frac{3}{2} P^{\mu\nu\rho\alpha} g^{\sigma\beta} + P^{\mu\rho\sigma\alpha} g^{\nu\beta} - P^{\mu\alpha\beta\rho} g^{\nu\sigma} \right] \delta_1 g_{\rho\sigma} \delta_2 g_{\alpha\beta}. \end{aligned} \quad (10.31)$$

Now, by adding the term “ $-d\mathbf{E}(g, \delta g, \mathcal{L}_\xi g)$ ” on both sides of the equation (10.24), one can derive a new fundamental identity for the Barnich-Compère symplectic current (10.30), if one redefines the surface charge (10.25) as

$$\mathbf{k}_\xi^{\text{BB}}(g, \delta g) \equiv \mathbf{k}_\xi^{\text{IW}}(g, \delta g) - \mathbf{E}(g, \delta g, \mathcal{L}_\xi g), \quad (10.32)$$

where  $\mathbf{k}_\xi^{\text{BB}}$  is called the *Barnich-Brandt surface charge*. Notice that for exact Killing vectors, *i.e.*,  $\mathcal{L}_\xi g = 0$ , the Iyer-Wald and Barnich-Brandt definitions of the surface charge are equivalent. In the rest of the paper, especially in appendix E, we restrict again to the Lee-Wald-Iyer proposals for  $\omega$  and  $\mathbf{k}_\xi$ .

## List of ambiguities

In the previous epigraphs we have given the “canonical” formulas for the relevant quantities in Wald’s formalism. However, these quantities are not uniquely defined. Let us here present a list of all the corresponding ambiguities. The symplectic potential  $\Theta$  and the Noether charge  $\mathbf{Q}_\xi$  are defined by (10.3) and (10.21), respectively, up to a closed

— and hence locally exact — form, denoted by  $d\mathbf{Y}$  and  $d\mathbf{Z}$ , respectively. Moreover, one can add a total derivative  $d\boldsymbol{\mu}$  to the Lagrangian without changing the equations of motion. These ambiguities  $\mathbf{Y}$ ,  $\mathbf{Z}$ , and  $\boldsymbol{\mu}$  also give rise to ambiguities in the other relevant quantities. The full list reads [41]

$$\mathbf{L} \rightarrow \mathbf{L} + d\boldsymbol{\mu}, \quad (10.33)$$

$$\boldsymbol{\Theta} \rightarrow \boldsymbol{\Theta} + \delta\boldsymbol{\mu} + d\mathbf{Y}(g, \delta g), \quad (10.34)$$

$$\boldsymbol{\omega} \rightarrow \boldsymbol{\omega} + d(\delta_1 \mathbf{Y}(g, \delta_2 g) - \delta_2 \mathbf{Y}(g, \delta_1 g)), \quad (10.35)$$

$$\mathbf{J}_\xi \rightarrow \mathbf{J}_\xi + d(\xi \cdot \boldsymbol{\mu}) + d\mathbf{Y}(g, \mathcal{L}_\xi g), \quad (10.36)$$

$$\mathbf{Q}_\xi \rightarrow \mathbf{Q}_\xi + \xi \cdot \boldsymbol{\mu} + \mathbf{Y}(g, \mathcal{L}_\xi g) + d\mathbf{Z}, \quad (10.37)$$

$$\mathbf{k}_\xi \rightarrow \mathbf{k}_\xi + \delta^{[g]} \mathbf{Y}(g, \mathcal{L}_\xi g) - \mathcal{L}_\xi \mathbf{Y}(g, \delta g) + d(\delta^{[g]} \mathbf{Z} + \xi \cdot \mathbf{Y}(g, \delta g)), \quad (10.38)$$

where the arrows mean that the expressions on the RHS are also compatible with the corresponding definitions. We have seen above that for exact Killing vectors the integral of  $\mathbf{k}_\xi$  is conserved. Moreover, here we observe that the integral of this form over a  $(D-2)$ -dimensional spacelike compact submanifold is unambiguous for Killing vectors, since in that case the total derivative does not contribute and we have [41]

$$\delta^{[g]} \mathbf{Y}(g, \mathcal{L}_\xi g) = \mathbf{Y}(g, \mathcal{L}_\xi \delta g) = \mathcal{L}_\xi \mathbf{Y}(g, \delta g), \quad (10.39)$$

because  $\mathcal{L}_\xi g = 0$ . Furthermore, we note that the Barnich-Compère symplectic current (10.30) and the Barnich-Brandt surface charge (10.32) do not fall within the class of ambiguities of the Wald definitions, (10.35) and (10.38), respectively. This is because the form  $\mathbf{E}(g, \delta_1 g, \delta_2 g)$  cannot be written as  $\delta_1 \mathbf{Y}(g, \delta_2 g) - \delta_2 \mathbf{Y}(g, \delta_1 g)$ , although it was previously suggested in [91] that this could be done. Thus, the proposals by Barnich-Brandt-Compère and Lee-Wald-Iyer for  $\boldsymbol{\omega}$  and  $\mathbf{k}_\xi$  are distinct, and it seems to depend on the problem which proposal is more appropriate<sup>36</sup>.

## 11 Final comments

In this paper we have presented a collection of new results on  $\mathcal{L}(\text{Riemann})$  theories of gravity. A summary of our findings can be found in section 1.1.

Before closing, we would like to point out that one of our motivations to study the linearized spectrum of this class of theories came from the following observations. In [19], the authors constructed a cubic theory admitting analytic extensions of the Schwarzschild-AdS black hole with a single blackening factor. Remarkably, they noticed

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<sup>36</sup>We thank Geoffrey Compère for clarifying this point.

that this theory — which was coined *Quasi-topological gravity*<sup>37</sup> — has the same linearized spectrum as Einstein gravity, *i.e.*, it falls in the Einstein-like category considered in section 3 — see appendix B. In fact, as far as we know, all the known examples of higher-order gravities<sup>38</sup> for which non-trivial analytic black-hole solutions — generalizing the corresponding Einstein gravity ones — have been constructed for generic values of the coupling<sup>39</sup> fall into the Einstein-like category: this includes Quasi-topological gravity [19, 61] and its generalizations to higher-curvatures *e.g.*, [95] and Lovelock theories [96–101]. In all those cases, if we restrict to static and spherically symmetric solutions — and analogously for planar or hyperbolic horizons — a single function determines the corresponding metric — *e.g.*, for Schwarzschild,  $f(r) = 1 - 2M/r$  in the usual coordinates. This is as opposed to black hole solutions of theories which do not belong to the Einstein-like class, *e.g.*, [105, 106], for which two independent functions are needed and generally can only be accessed numerically or in certain limits. This suggests the possibility of finding simple analytic extensions of Einstein’s gravity black holes for that class of theories. Furthermore, it is natural to expect that only theories that do not propagate the extra scalar and the ghost-like graviton at the linearized level are susceptible of admitting extensions of Schwarzschild’s solution with a single blackening factor. Additional evidence in favor of these claims coming from ECG was recently reported in [29, 102–104]. A general study for arbitrary  $\mathcal{L}(\text{Riemann})$  theories is also in progress.

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<sup>37</sup>Note that, as opposed to *e.g.*, ECG, Quasi-topological gravity is defined in a dimension-dependent fashion.

<sup>38</sup>In this statement we are referring to purely gravitational metric theories.

<sup>39</sup>The situation changes if one allows for fine-tuned couplings — see *e.g.*, [62, 92, 93]. Another possibility is considering theories which do not reduce to Einstein gravity when the corresponding couplings vanish, like pure  $R^2$  gravity, *e.g.*, [94]. We find these set-ups considerably less interesting.

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## A Linearization procedure: examples

In this appendix we apply the linearization procedure explained in section 2 to two instances. The first is a general quadratic theory in  $D$ -dimensions, for which we give details of all the steps involved in the linearization process. The second is a Born-Infeld gravity. Our goal in that case is to illustrate that our method can be easily applied to theories whose linearization would be difficult to achieve using different methods.

### Quadratic gravity

Let us consider the most general quadratic gravity in general dimensions,

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} (\alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho}) \right\}. \quad (\text{A.1})$$

In order to obtain  $\mathcal{L}(\Lambda, \alpha)$ , we only have to substitute the Riemann tensors appearing in the above Lagrangian density by the expression (2.20) and use the algebraic properties of the auxiliary tensor  $k_{\mu\nu}$  (2.19) to compute all the contractions. We find

$$\begin{aligned} R^2 \Big|_{(\Lambda, \alpha)} &= \Lambda^2 D^2 (D-1)^2 + 2\Lambda\alpha D(D-1)\chi(\chi-1) + \alpha^2 \chi^2 (\chi-1)^2, \\ R_{\mu\nu} R^{\mu\nu} \Big|_{(\Lambda, \alpha)} &= \Lambda^2 D(D-1)^2 + 2\Lambda\alpha(D-1)\chi(\chi-1) + \alpha^2 \chi(\chi-1)^2, \\ R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho} \Big|_{(\Lambda, \alpha)} &= 2D(D-1)\Lambda^2 + 4\Lambda\alpha\chi(\chi-1) + 2\alpha^2 \chi(\chi-1). \end{aligned} \quad (\text{A.2})$$

The final result for  $\mathcal{L}(\Lambda, \alpha)$  reads

$$\begin{aligned} \mathcal{L}(\Lambda, \alpha) &= + \frac{1}{2\kappa} (-2\Lambda_0 + \Lambda D(D-1) + \alpha\chi(\chi-1)) \\ &+ \kappa^{\frac{4-D}{D-2}} (\Lambda^2 D(D-1) + 2\Lambda\alpha\chi(\chi-1)) (D(D-1)\alpha_1 + (D-1)\alpha_2 + 2\alpha_3) \\ &+ \kappa^{\frac{4-D}{D-2}} \alpha^2 \chi(\chi-1) (\chi(\chi-1)\alpha_1 + (\chi-1)\alpha_2 + 2\alpha_3). \end{aligned} \quad (\text{A.3})$$

Then, applying (2.22) we get

$$e = \frac{1}{4\kappa} + \Lambda \kappa^{\frac{4-D}{D-2}} (D(D-1)\alpha_1 + (D-1)\alpha_2 + 2\alpha_3). \quad (\text{A.4})$$

The second derivative with respect to  $\alpha$  yields

$$\frac{\partial^2 \mathcal{L}}{\partial \alpha^2} = 2\chi(\chi - 1)\kappa^{\frac{4-D}{D-2}} \left[ \chi(\chi - 1)\alpha_1 + (\chi - 1)\alpha_2 + 2\alpha_3 \right]. \quad (\text{A.5})$$

Hence, comparing with (2.23) we can easily obtain the values of  $a$ ,  $b$  and  $c$ . The result is

$$a = \kappa^{\frac{4-D}{D-2}} \alpha_3, \quad b = \frac{\kappa^{\frac{4-D}{D-2}} \alpha_1}{2}, \quad c = \frac{\kappa^{\frac{4-D}{D-2}} \alpha_2}{2}. \quad (\text{A.6})$$

Inserting the values of  $a$ ,  $b$ ,  $c$  and  $e$  into (2.28)-(2.30) gives rise to equations (2.32)-(2.34) for  $\kappa_{\text{eff}}$ ,  $m_s^2$  and  $m_g^2$ .

Finally, from (2.14) we see that the cosmological constant is related to the background scale  $\Lambda$  and the couplings of the theory through

$$\Lambda_0 = \frac{(D-1)(D-2)\Lambda}{2} + \kappa^{\frac{2}{D-2}} \Lambda^2 (D-4)(D-1) [D(D-1)\alpha_1 + (D-1)\alpha_2 + 2\alpha_3]. \quad (\text{A.7})$$

### Born-Infeld gravity

Let us now consider the following theory, which has the form of a Born-Infeld model

$$S = \frac{1}{\kappa^{\frac{D}{D-2}} (1+\lambda)^{\frac{D-2}{2}}} \int_{\mathcal{M}} d^D x \left[ \sqrt{|g_{\mu\nu}(1+\lambda) + \kappa^{\frac{2}{D-2}} R_{\mu\nu}|} - \sqrt{|g_{\mu\nu}|} \right], \quad (\text{A.8})$$

where  $|A_{\mu\nu}|$  stands for the absolute value of the determinant and  $\lambda$  is a dimensionless parameter — which we assume to be greater than  $-1$ . The normalization is chosen so that to leading order the action becomes Einstein-Hilbert

$$S = \frac{1}{2\kappa} \int_{\mathcal{M}} d^D x \sqrt{|g|} \left[ -2\Lambda_0 + R + \dots \right], \quad (\text{A.9})$$

where  $\Lambda_0 = [(1+\lambda)^{1-D/2} - (1+\lambda)] \kappa^{\frac{2}{2-D}}$ , and the ellipsis mean an infinite series of higher order terms in curvature. Linearizing this theory can be a non-trivial task, due to the presence of the determinant and the square root. Using our method, it becomes quite easy though. First, extracting as common factor the square root of the metric determinant<sup>40</sup>, we find the Lagrangian density

$$\kappa^{\frac{D}{D-2}} (1+\lambda)^{\frac{D-2}{2}} \mathcal{L} = \sqrt{|(1+\lambda)\delta^\mu_\nu + \kappa^{\frac{2}{D-2}} R^\mu_\nu|} - 1. \quad (\text{A.10})$$

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<sup>40</sup>We use that  $|A_{\mu\nu}| = |g_{\mu\alpha} A^\alpha_\nu| = |g_{\mu\nu}| |A^\alpha_\beta|$ .



Now, we follow our recipe and substitute the ‘‘Riemann tensor’’ (2.20) in this expression

$$\kappa^{\frac{D}{D-2}}(1+\lambda)^{\frac{D-2}{2}}\mathcal{L}(\Lambda, \alpha) = \sqrt{\left| \left( 1 + \lambda + \kappa^{\frac{2}{D-2}}\Lambda(D-1) \right) \delta^\mu_\nu + \alpha \kappa^{\frac{2}{D-2}}(\chi-1)k^\mu_\nu \right|} - 1. \quad (\text{A.11})$$

The determinant can be computed using (2.19) and the identity

$$|A| = e^{\text{tr}(\log A)}. \quad (\text{A.12})$$

The result is

$$\kappa^{\frac{D}{D-2}}(1+\lambda)^{\frac{D-2}{2}}\mathcal{L}(\Lambda, \alpha) = \left( 1 + \lambda + \kappa^{\frac{2}{D-2}}\Lambda(D-1) \right)^{D/2} \left( 1 + \frac{\alpha \kappa^{\frac{2}{D-2}}(\chi-1)}{1 + \lambda + \kappa^{\frac{2}{D-2}}\Lambda(D-1)} \right)^{\chi/2} - 1. \quad (\text{A.13})$$

This ‘‘prepotential’’ contains all the information about the linearized theory. Let us begin by determining  $\Lambda$ . The equation for the background curvature (2.14) becomes

$$\left[ 1 + \lambda + \kappa^{\frac{2}{D-2}}\Lambda(D-1) \right]^{D/2} - 1 = \kappa^{\frac{2}{D-2}}\Lambda(D-1) \left[ 1 + \lambda + \kappa^{\frac{2}{D-2}}\Lambda(D-1) \right]^{D/2-1}. \quad (\text{A.14})$$

A simple algebraic manipulation yields

$$1 = (1 + \lambda) \left[ 1 + \lambda + \kappa^{\frac{2}{D-2}}\Lambda(D-1) \right]^{D/2-1}. \quad (\text{A.15})$$

Thus, since we have assumed  $\lambda > -1$ , this equation has always one solution:

$$\Lambda = \frac{1}{\kappa^{\frac{2}{D-2}}(D-1)} \left[ (1 + \lambda)^{-2/(D-2)} - (1 + \lambda) \right]. \quad (\text{A.16})$$

Now we can compute the parameters  $a, b, c$  and  $e$ . From (2.22) we get

$$e = \frac{1}{4\kappa} (1 + \lambda)^{-D/2}, \quad (\text{A.17})$$

where we already evaluated the expression on the background. On the other hand, the second derivative of  $\mathcal{L}(\Lambda, \alpha)$  with respect to  $\alpha$  evaluated at  $\alpha = 0$  yields

$$\frac{1}{4\chi(\chi-1)} \frac{\partial^2 \mathcal{L}}{\partial \alpha^2} \Big|_{\alpha=0} = \frac{1}{16} \kappa^{\frac{4-D}{D-2}} (\chi-1)(\chi-2)(1+\lambda)^{-\frac{D^2-2D-4}{2(D-2)}}, \quad (\text{A.18})$$

where we have also made use of (A.16). Now, comparing this expression with (2.23), we find the values of the parameters, namely

$$a = 0, \quad b = \frac{1}{16} \kappa^{\frac{4-D}{D-2}} (1 + \lambda)^{-\frac{D^2-2D-4}{2(D-2)}}, \quad c = -\frac{1}{8} \kappa^{\frac{4-D}{D-2}} (1 + \lambda)^{-\frac{D^2-2D-4}{2(D-2)}}. \quad (\text{A.19})$$

Finally, using (2.28)-(2.30) we can compute the physical parameters  $\kappa_{\text{eff}}$ ,  $m_s$  and  $m_g$

$$\kappa_{\text{eff}} = \kappa(1 + \lambda)^{D/2}, \quad m_s^2 = 2(1 + \lambda)\kappa^{\frac{2}{2-D}}, \quad m_g^2 = 2(1 + \lambda)^{-2/(D-2)}\kappa^{\frac{2}{2-D}}. \quad (\text{A.20})$$

Therefore, we have completely characterized the linearized spectrum of this Born-Infeld model. Since we assumed that  $\lambda > -1$ , all quantities are finite and real, and everything is well-defined. For  $D > 2$ , the background (A.16) is dS ( $\Lambda > 0$ ) when  $\lambda < 0$ , AdS ( $\Lambda < 0$ ) when  $\lambda > 0$  and flat when  $\lambda = 0$ . In all cases we have, apart from the massless graviton, a massive scalar and a massive spin-2 graviton. The masses squared and the effective gravitational constant are always positive.

## B Classification of theories: examples

In this appendix we provide numerous examples of the different classes of theories characterized in section 3.

### Theories without massive graviton

In section 7 we characterized all theories being defined in a dimension-independent manner which do not propagate the extra massive graviton up to cubic order in curvature. The list of theories reduced to the particular  $f(\text{Lovelock})$  terms, ECG (6.2) plus a new invariant,  $\mathcal{Y}$ , which we defined in (7.6). In this appendix we will study general  $f(\text{Lovelock})$  theories, which — although not necessarily defined in a dimension-independent way — are a paradigmatic example of theories which only propagate the usual massless graviton plus the scalar at the linearized level [62].

#### $f(\text{Lovelock})$ gravities

The most general  $f(\text{Lovelock})$  action can be written as

$$S = \frac{1}{2\kappa} \int_{\mathcal{M}} d^D x \sqrt{|g|} f(\mathcal{L}_0, \mathcal{L}_1, \dots, \mathcal{L}_{\lfloor D/2 \rfloor}), \quad (\text{B.1})$$

where  $f$  is some differentiable function of the dimensionally-extended Euler densities<sup>41</sup>

$$\mathcal{L}_k \equiv \frac{1}{2^k} \delta_{\alpha_1 \beta_1 \dots \alpha_k \beta_k}^{\mu_1 \nu_1 \dots \mu_k \nu_k} R_{\mu_1 \nu_1}^{\alpha_1 \beta_1} \dots R_{\mu_k \nu_k}^{\alpha_k \beta_k}, \quad (\text{B.2})$$

where the generalized Kronecker symbol is defined as  $\delta_{\alpha_1 \beta_1 \dots \alpha_k \beta_k}^{\mu_1 \nu_1 \dots \mu_k \nu_k} \equiv (2k)! \delta_{\alpha_1}^{\mu_1} \delta_{\beta_1}^{\nu_1} \dots \delta_{\alpha_k}^{\mu_k} \delta_{\beta_k}^{\nu_k}$ . Note that the first three densities are nothing but: a constant that can be identified with

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<sup>41</sup>Namely,  $\mathcal{L}_k$  becomes the Euler density when evaluated for a  $2k$ -dimensional manifold.

the cosmological constant  $\mathcal{L}_0 \equiv -2\Lambda_0$ ; the Einstein-Hilbert term,  $\mathcal{L}_1 \equiv R$ ; and Gauss-Bonnet gravity,  $\mathcal{L}_2 \equiv \mathcal{X}_4$ . A corollary from the results presented in section 5 is that  $f(\text{Lovelock})$  theories inherit the property of Lovelock gravities of not propagating the massive graviton<sup>42</sup>. This means that the linearized equations of motion for  $f(\text{Lovelock})$  gravities should not involve the  $\bar{\square} G_{\mu\nu}^L$  term. This is indeed the case. In particular, they read [62]

$$\mathcal{E}_{\mu\nu}^L = \alpha G_{\mu\nu}^L + \Lambda \beta \bar{g}_{\mu\nu} R^L + \frac{\beta}{D-1} (\bar{g}_{\mu\nu} \bar{\square} - \bar{\nabla}_\nu \bar{\nabla}_\mu) R^L = 0, \quad (\text{B.3})$$

where  $\alpha$  and  $\beta$  are the following constants<sup>43</sup>

$$\alpha \equiv \frac{1}{2\kappa} \sum_{k=1}^{\lfloor D/2 \rfloor} \partial_k f(\bar{\mathcal{L}}) \frac{k(D-3)!}{(D-2k-1)!} \Lambda^{k-1}, \quad (\text{B.4})$$

$$\beta \equiv \frac{1}{2\kappa} \sum_{k,l=1}^{\lfloor D/2 \rfloor} \partial_k \partial_l f(\bar{\mathcal{L}}) \frac{kl(D-2)!(D-1)!}{(D-2k)!(D-2l)!} \Lambda^{k+l-2}. \quad (\text{B.5})$$

Here  $\partial_l f(\bar{\mathcal{L}})$  means that we should take a formal derivative of  $f$  with respect to the corresponding dimensionally-extended Euler density, and then evaluate the result in the background. Comparing with the linearized equations (3.2), we see that  $\alpha$  determines the effective Einstein constant  $\kappa_{\text{eff}}$  and  $\beta$  is related to the mass of the scalar field

$$\kappa_{\text{eff}} = \frac{1}{2\alpha}, \quad m_s^2 = \frac{D-2-2\beta D\Lambda}{2\beta}. \quad (\text{B.6})$$

Note that for  $\beta = 0$  the scalar mode is also absent, and the only physical field is the massless graviton. This applies *e.g.*, to pure Lovelock gravities, but also to other non-trivial theories [62] — some of which we review in the last epigraph of this section. The parameters  $a, b, c$  and  $e$  are given by

$$a = -\frac{1}{2}c = -\frac{\alpha - 2e}{4(D-3)\Lambda}, \quad b = \frac{\beta}{4(D-1)} - \frac{\alpha - 2e}{8(D-3)\Lambda}, \quad e = \frac{f(\bar{\mathcal{L}})}{8\kappa\Lambda(D-1)}, \quad (\text{B.7})$$

and the background embedding equation (2.14) reads in turn

$$f(\bar{\mathcal{L}}) = \sum_{k=1}^{\lfloor D/2 \rfloor} \frac{2k(D-1)!}{(D-2k)!} \Lambda^k \partial_k f(\bar{\mathcal{L}}). \quad (\text{B.8})$$

<sup>42</sup>In appendix C we show how the linearized equations of  $f(R)$  can be obtained from those of Einstein gravity. The procedure can be naturally applied as well to  $f(\text{Lovelock})$  theories starting from Lovelock, and the results will match the ones presented in this appendix.

<sup>43</sup>Note that  $\lfloor D/2 \rfloor$  stands for the largest integer smaller or equal to  $D/2$ .

## Theories without dynamical scalar

### Conformal gravity

In the case of quadratic gravity, the most general theory which does not propagate a scalar field is [108]

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} \left[ \beta \left( R^2 - \frac{4(D-1)}{D} R_{\mu\nu} R^{\mu\nu} \right) + \gamma \mathcal{X}_4 \right] \right\}, \quad (\text{B.9})$$

where  $\mathcal{X}_4$  is again the Gauss-Bonnet term and  $\beta$  and  $\gamma$  are dimensionless constants. Observe that for  $D = 3$ , this action is equivalent to *new massive gravity* [109]. There are two different interesting ways of writing this theory in terms of other well-known curvature tensors. Firstly, it was observed in [110] that the contraction of the Einstein tensor  $G_{\mu\nu}$  with the Schouten tensor<sup>44</sup>  $S_{\mu\nu}$  is proportional to the curvature invariant in (B.9) that multiplies  $\beta$

$$G_{\mu\nu} S^{\mu\nu} = -\frac{D}{4(D-2)(D-1)} \left( R^2 - \frac{4(D-1)}{D} R_{\mu\nu} R^{\mu\nu} \right). \quad (\text{B.10})$$

Therefore, by rescaling  $\beta$  we see that the theory is equivalent to

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} (\tilde{\beta} G_{\mu\nu} S^{\mu\nu} + \gamma \mathcal{X}_4) \right\}. \quad (\text{B.11})$$

Secondly, it turns out that the quadratic part of (B.9) is equivalent to the higher dimensional version of conformal gravity, consisting of the square of the Weyl tensor, together with a Gauss-Bonnet term. The square of the Weyl tensor is in fact equal to<sup>45</sup>

$$C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = \mathcal{X}_4 - \frac{D(D-3)}{(D-2)(D-1)} \left( R^2 - \frac{4(D-1)}{D} R_{\mu\nu} R^{\mu\nu} \right). \quad (\text{B.12})$$

By using this relation and redefining the couplings, the theory can be written as

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} (\tilde{\beta} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} + \tilde{\gamma} \mathcal{X}_4) \right\}. \quad (\text{B.13})$$

Thus, we observe that conformal gravity in any dimension is free of the scalar mode, and only propagates the two gravitons. Finally, for this theory the effective gravitational

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<sup>44</sup>The Schouten tensor is defined as  $S_{\mu\nu} \equiv \frac{1}{D-2} \left( R_{\mu\nu} - \frac{1}{2(D-1)} R g_{\mu\nu} \right)$ .

<sup>45</sup>The Weyl tensor is defined as  $C_{\mu\nu\rho\sigma} \equiv R_{\mu\nu\rho\sigma} - \frac{2}{D-2} (g_{\mu[\rho} R_{\sigma]\nu} - g_{\nu[\rho} R_{\sigma]\mu}) + \frac{2}{(D-1)(D-2)} R g_{\mu[\rho} g_{\sigma]\nu}$ .

constant and the mass of the extra graviton read respectively

$$\kappa_{\text{eff}} = \frac{\kappa}{1 - 4\kappa^{\frac{2}{D-2}}\Lambda(D-3)(2\tilde{\beta} - \tilde{\gamma}(D-4))}, \quad (\text{B.14})$$

$$m_g^2 = \frac{2 - D + 4\kappa^{\frac{2}{D-2}}\Lambda(D-3)(D-2)(2\tilde{\beta} - \tilde{\gamma}(D-4))}{8\tilde{\beta}\kappa^{\frac{2}{D-2}}(D-3)}. \quad (\text{B.15})$$

If the numerator of (B.15) becomes zero, then the extra graviton is massless. This particular case will be analyzed in the epigraph on critical gravities. Note finally that in  $D = 3$  both the Weyl tensor and the Gauss-Bonnet term vanish identically, so the theory reduces to Einstein gravity plus cosmological constant.

## Theories with two massless gravitons

The following is an example of a theory propagating two massless gravitons in addition to the scalar field,

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} \alpha R^2 - D \left( \kappa^{\frac{4-D}{D-2}} \alpha + \frac{1}{16\kappa\Lambda_0} \right) R_{\mu\nu} R^{\mu\nu} \right\}. \quad (\text{B.16})$$

Note that the  $m_g^2 = 0$  condition has the unpleasant feature of mixing the couplings of terms of different order in curvature. In this case, we see that the  $R_{\mu\nu} R^{\mu\nu}$  coupling depends on the combination  $\kappa\Lambda_0$ . For this theory, the background scale is related to the cosmological constant by

$$\Lambda = \frac{4\Lambda_0}{D(D-1)}. \quad (\text{B.17})$$

In addition, the effective gravitational constant and the mass of the scalar field read

$$\hat{\kappa}_{\text{eff}} = \frac{2(D-1)\kappa\Lambda}{1 + 4\Lambda\kappa^{\frac{2}{D-2}}\alpha D(D-1)}, \quad m_s^2 = -\frac{4(D-1)\Lambda}{D + 4\Lambda\kappa^{\frac{2}{D-2}}\alpha(D-1)(D-2)^2}. \quad (\text{B.18})$$

As far as we know, this theory has not been considered before.

## Critical gravities

*Critical gravity* was introduced in [27] as the four-dimensional quadratic theory for which the extra graviton is massless and the scalar mode is absent. Hence, it is a special case of the theories considered in the last two epigraphs — (B.9) and (B.16) — in the particular case of  $D = 4$ . The following action is a generalization of critical gravity to general

dimensions [110]

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) - \frac{D^2}{16\kappa\Lambda_0(D-2)^2} \left( R^2 - \frac{4(D-1)}{D} R_{\mu\nu} R^{\mu\nu} \right) \right\}. \quad (\text{B.19})$$

It can be obtained by setting  $\beta = -D^2/(16\kappa^{2/(D-2)}\Lambda_0(D-2)^2)$  and  $\gamma = 0$  in (B.9) or, alternatively, from (B.16) if we put  $\alpha = -D^2/(16\kappa^{2/(D-2)}\Lambda_0(D-2)^2)$  there. In  $D = 4$ , this is the critical theory considered by [27], and for  $D = 3$ , it is equivalent to *critical new massive gravity* with a cosmological constant [111]. Furthermore, the effective gravitational constant of this theory is

$$\hat{\kappa}_{\text{eff}} = -\frac{1}{2}(D-2)^2\kappa\Lambda, \quad (\text{B.20})$$

which is only positive for  $\Lambda < 0$ .

## Einstein-like theories

In section 6 we already constructed examples of *Einstein-like* theories in the sense defined in section 3, *i.e.*, theories which only propagate a massless graviton on a msb. However, the theories considered in that section had the additional property of being defined in a dimension-independent manner and we coined them *Einsteinian*. In this appendix we would like to present some more examples of *Einstein-like* theories whose definition does however depend on the spacetime dimension.

### Quasi-topological gravity

The first example is *Quasi-topological gravity* [18, 19, 61]. This is a cubic theory which has the nice property of admitting analytic black hole solutions — which generalize Schwarzschild-AdS and its Gauss-Bonnet generalization [99]. It consists of a combination of all Lovelock gravities up to cubic order plus an additional “Quasi-topological” term:

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{4-D}{D-2}} \alpha \mathcal{X}_4 + \kappa^{\frac{6-D}{D-2}} [\beta \mathcal{X}_6 + \gamma \mathcal{Z}] \right\}. \quad (\text{B.21})$$

Here the cubic Lovelock term is given by

$$\begin{aligned} \mathcal{X}_6 \equiv & -8R_{\mu}^{\rho} R_{\nu}^{\sigma} R_{\rho}^{\delta} R_{\sigma}^{\gamma} R_{\delta}^{\mu} R_{\gamma}^{\nu} + 4R_{\mu\nu}^{\rho\sigma} R_{\rho\sigma}^{\delta\gamma} R_{\delta\gamma}^{\mu\nu} - 24R_{\mu\nu\rho\sigma} R^{\mu\nu\rho}_{\delta} R^{\sigma\delta} \\ & + 3R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} R + 24R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + 16R_{\mu}^{\nu} R_{\nu}^{\rho} R_{\rho}^{\mu} - 12R_{\mu\nu} R^{\mu\nu} R + R^3, \end{aligned} \quad (\text{B.22})$$

and the Quasi-topological one in general dimensions reads in turn [19]

$$\begin{aligned}\mathcal{Z} \equiv & R_{\mu}^{\rho} R_{\nu}^{\sigma} R_{\rho}^{\delta} R_{\sigma}^{\gamma} R_{\delta}^{\mu} R_{\gamma}^{\nu} + \frac{1}{(2D-3)(D-4)} \left( -3(D-2) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho}_{\delta} R^{\sigma\delta} \right. \\ & + \frac{3(3D-8)}{8} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} R + 3D R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} \\ & \left. + 6(D-2) R_{\mu}^{\nu} R_{\nu}^{\rho} R_{\rho}^{\mu} - \frac{3(3D-4)}{2} R_{\mu\nu} R^{\mu\nu} R + \frac{3D}{8} R^3 \right). \end{aligned} \quad (\text{B.23})$$

The physical quantities for (B.21) read

$$\kappa_{\text{eff}} = \frac{\kappa}{f(\alpha, \beta, \gamma, \Lambda, \kappa)}, \quad m_s = +\infty, \quad m_g = +\infty, \quad (\text{B.24})$$

where

$$\begin{aligned}f(\alpha, \beta, \gamma, \Lambda, \kappa) \equiv & +1 + 4\Lambda\kappa^{\frac{2}{D-2}}\alpha(D-4)(D-3) \\ & + 6\kappa^{\frac{4}{D-2}}\Lambda^2\beta(D-6)(D-5)(D-4)(D-3) \\ & + \frac{3(D-6)(D-3)}{4(2D-3)}\kappa^{\frac{4}{D-2}}\Lambda^2\gamma(16+3D(D-5)). \end{aligned}$$

Hence, as explained in [19], this theory shares the linearized spectrum of Einstein gravity.

### Special $f(\text{Lovelock})$ theories

The second example we would like to consider corresponds to a particular family of  $f(\text{Lovelock})$  gravities. As we explained before, all  $f(\text{Lovelock})$  theories are free of the massive graviton, but do in general propagate the extra scalar. However, as pointed out in [62] it is possible to construct non-trivial theories — *i.e.*, different from the pure Lovelock case — which are also free of the extra scalar and hence share the linearized spectrum of Einstein gravity.

Indeed, whenever  $\beta$ , as defined in (B.5), vanishes, the mass of the scalar diverges — which is obvious from (B.6). This is achieved whenever  $\partial_k \partial_l f(\bar{\mathcal{L}}) = 0$  for all  $k, l$ , which leaves us with nothing but the usual Lovelock theory or, alternatively, if

$$\sum_{k,l=1}^{[D/2]} \partial_k \partial_l f(\bar{\mathcal{L}}) \frac{kl(D-2)!(D-1)!}{(D-2k)!(D-2l)!} \Lambda^{k+l-2} = 0, \quad \partial_k \partial_l f(\bar{\mathcal{L}}) \neq 0, \quad (\text{B.25})$$

for some  $k, l$ . This equation is *e.g.*, satisfied by all theories of the form [62]

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{2(u+2s)-D}{D-2}} \lambda (R^u \mathcal{L}_2^s - \gamma R^{2s+u}) \right\}, \quad (\text{B.26})$$

where  $\gamma$  is the dimension-dependent constant

$$\gamma \equiv \frac{u^2 + 4(s-1)s + u(4s-1)}{(u+2s)(u+2s-1)} \frac{(D-2)^s (D-3)^s}{D^s (D-1)^s}, \quad (\text{B.27})$$

for any  $u, s \geq 0$ . In particular, for  $s = u = 1$  one finds the cubic class of theories

$$S = \int_{\mathcal{M}} d^D x \sqrt{|g|} \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \kappa^{\frac{6-D}{D-2}} \lambda \left[ R\mathcal{L}_2 - \left( \frac{2(D-2)(D-3)}{3D(D-1)} \right) R^3 \right] \right\}. \quad (\text{B.28})$$

The  $D = 4$  case of (B.28) was also considered in [28] in a slightly different context. The effective gravitational constant of (B.28) reads

$$\kappa_{\text{eff}} = \kappa \left[ 1 + 2(D-6)(D-3)(D-1)D\lambda\kappa^{\frac{4}{D-2}}\Lambda^2 \right]^{-1}. \quad (\text{B.29})$$

## C $f(\text{scalars})$ theories: examples

Let us now illustrate how the expressions obtained in section 5 can be used to easily compute the values of  $a, b, c$  and  $e$  for theories consisting of functions of invariants, as long as we know the values of those parameters for the invariants themselves.

### $f(R)$ gravity

Let us first consider  $f(R)$  gravity, whose Lagrangian in general dimensions we write as

$$S = \frac{1}{2\kappa} \int_{\mathcal{M}} d^D x \sqrt{|g|} f(R). \quad (\text{C.1})$$

According to table 2, for  $R$  we have  $a = b = c = 0$ ,  $e = \frac{1}{2}$  and  $\bar{R} = D(D-1)\Lambda$ . Therefore, using the transformation rules (5.4) for the theory above we have

$$a = c = 0, \quad b = \frac{1}{8\kappa} f''(\bar{R}), \quad e = \frac{1}{4\kappa} f'(\bar{R}). \quad (\text{C.2})$$

Note that these expressions can also be easily obtained from the general  $f(\text{Lovelock})$  ones (B.7). Also, according to (2.14) the background curvature  $\Lambda$  is determined by the equation

$$f(\bar{R}) = 2(D-1)\Lambda f'(\bar{R}). \quad (\text{C.3})$$

If  $f''(\bar{R}) \neq 0$ , we have a scalar mode with mass

$$m_s^2 = \frac{(D-2)f'(\bar{R}) - 2\bar{R}f''(\bar{R})}{2(D-1)f''(\bar{R})}. \quad (\text{C.4})$$

The effective gravitational constant is in turn given by

$$\kappa_{\text{eff}} = \frac{\kappa}{f'(\bar{R})}. \quad (\text{C.5})$$



$f(R, R_{\mu\nu}^2, R_{\mu\nu\rho\sigma}^2)$  **gravity**

Let us now study all theories that can be constructed as functions of invariants up to quadratic order [112]. The independent scalars are  $R$ ,  $Q \equiv R_{\mu\nu}R^{\mu\nu}$ , and  $K \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ , so let us consider an action of the form

$$S = \frac{1}{2\kappa} \int_{\mathcal{M}} d^D x \sqrt{|g|} f(R, Q, K). \quad (\text{C.6})$$

This theory includes, as particular cases,  $f(R)$  and general quadratic gravities. In order to simplify the following expressions, let us write  $\bar{\mathcal{R}} \equiv (R, Q, K)$ . Evaluated on the background, the invariants read

$$\bar{\mathcal{R}} = (D(D-1)\Lambda, D(D-1)^2\Lambda^2, 2D(D-1)\Lambda^2). \quad (\text{C.7})$$

Then, the background embedding equation (2.14) can be written in terms of these background scalars  $\bar{R}, \bar{Q}, \bar{K}$  as

$$\bar{R}\partial_R f(\bar{\mathcal{R}}) + 2\bar{Q}\partial_Q f(\bar{\mathcal{R}}) + 2\bar{K}\partial_K f(\bar{\mathcal{R}}) = \frac{D}{2} f(\bar{\mathcal{R}}), \quad (\text{C.8})$$

which, in particular, generalizes (C.3) for this theory. Finally, the parameters  $a, b, c$  and  $e$  are given by

$$\begin{aligned} a &= \frac{1}{2\kappa} \partial_K f(\bar{\mathcal{R}}), \\ b &= \frac{1}{2\kappa} \left[ \frac{1}{4} \partial_R \partial_R f(\bar{\mathcal{R}}) + (D-1)\Lambda \partial_R \partial_Q f(\bar{\mathcal{R}}) + 2\Lambda \partial_R \partial_K f(\bar{\mathcal{R}}) \right. \\ &\quad \left. + (D-1)^2 \Lambda^2 \partial_Q \partial_Q f(\bar{\mathcal{R}}) + 4(D-1)\Lambda^2 \partial_Q \partial_K f(\bar{\mathcal{R}}) + 4\Lambda^2 \partial_K \partial_K f(\bar{\mathcal{R}}) \right], \\ c &= \frac{1}{4\kappa} \partial_Q f(\bar{\mathcal{R}}), \\ e &= \frac{1}{4\kappa} [\partial_R f(\bar{\mathcal{R}}) + 2(D-1)\Lambda \partial_Q f(\bar{\mathcal{R}}) + 4\Lambda \partial_K f(\bar{\mathcal{R}})], \end{aligned} \quad (\text{C.9})$$

from which one can easily obtain the values of  $\kappa_{\text{eff}}, m_s^2, m_g^2$ .

## D Einsteinian quartic gravities

Here we provide the explicit expressions for the conditions  $F_g^{(2)}(\alpha_i) = F_s^{(2)}(\alpha_i) = F_g^{(3)}(\beta_i, D) = F_s^{(3)}(\beta_i, D) = F_g^{(4)}(\gamma_i, D) = F_s^{(4)}(\gamma_i, D) = 0$  appearing in section 6. These

read:

$$F_g^{(2)}(\alpha_i) \equiv +\frac{1}{2}\alpha_2 + 2\alpha_3 = 0, \quad (\text{D.1})$$

$$F_s^{(2)}(\alpha_i) \equiv +2\alpha_1 + \frac{1}{2}\alpha_2 = 0, \quad (\text{D.2})$$

$$F_g^{(3)}(\beta_i, D) \equiv -\frac{3}{2}\beta_1 + 12\beta_2 + 2D\beta_3 + 2D(D-1)\beta_4 + \left(D - \frac{3}{2}\right)\beta_5 \\ + \frac{3}{2}(D-1)\beta_6 + \frac{1}{2}D(D-1)\beta_7 = 0, \quad (\text{D.3})$$

$$F_s^{(3)}(\beta_i, D) \equiv +\frac{3}{2}\beta_1 + 2\beta_3 + 8\beta_4 + \left(D + \frac{1}{2}\right)\beta_5 + \frac{3}{2}(D-1)\beta_6 \\ + (D-1)\left(\frac{D}{2} + 4\right)\beta_7 + 6D(D-1)\beta_8 = 0, \quad (\text{D.4})$$

$$F_g^{(4)}(\gamma_i, D) \equiv + (4D-9)\gamma_1 + 2(D+3)\gamma_2 + (2D-9)\gamma_3 + 24\gamma_4 + 48\gamma_5 + 8\gamma_6 \quad (\text{D.5}) \\ + 4D(D-1)\gamma_7 - \frac{1}{2}(D+3)\gamma_8 + 6(2D-1)\gamma_9 + (2D^2 - D - 3)\gamma_{10} \\ - \frac{3}{2}D(D-1)\gamma_{11} + 12D(D-1)\gamma_{12} + \left(2D^2 + \frac{1}{2}D - 3\right)\gamma_{13} \\ + \frac{1}{2}(3D^2 - 8D + 6)\gamma_{14} + (2D^2 - 3)\gamma_{15} + (2D^2 + D - 3)\gamma_{16} \\ + D(D-1)(2D-1)\gamma_{17} + 2D^2(D-1)\gamma_{18} + 2D^2(D-1)^2\gamma_{19} \\ + (D-1)(2D-3)\gamma_{20} + \frac{1}{2}D(D-1)(2D-3)\gamma_{21} + 3(D-1)^2\gamma_{22} \\ + D(D-1)^2\gamma_{23} + \frac{3}{2}D(D-1)^2\gamma_{24} + \frac{1}{2}D^2(D-1)^2\gamma_{25} = 0.$$

$$\begin{aligned}
F_s^{(4)}(\gamma_i, D) \equiv & +7\gamma_1 + 2(D-1)\gamma_2 + 5\gamma_3 + 8\gamma_6 + 32\gamma_7 + \frac{5}{2}(D-1)\gamma_8 \\
& + 6\gamma_9 + 3(D-1)\gamma_{10} + \frac{3}{2}(D^2 + 3D - 8)\gamma_{11} + 24\gamma_{12} + \frac{3}{2}(3D-2)\gamma_{13} \\
& + \frac{1}{2}(3D^2 + 4D - 10)\gamma_{14} + (4D-1)\gamma_{15} + 5(D-1)\gamma_{16} \\
& + (D+16)(D-1)\gamma_{17} + 2(D+6)(D-1)\gamma_{18} + 20D(D-1)\gamma_{19} \\
& + (D-1)(2D+1)\gamma_{20} + \frac{1}{2}(D-1)(2D^2 + 13D - 12)\gamma_{21} \\
& + 3(D-1)^2\gamma_{22} + (D-1)^2(D+8)\gamma_{23} + \frac{3}{2}(D-1)^2(D+4)\gamma_{24} \\
& + \frac{1}{2}D(D-1)^2(D+20)\gamma_{25} + 12D^2(D-1)^2\gamma_{26} = 0.
\end{aligned} \tag{D.6}$$

Solving the last two equations order by order in  $D$  gives rise to the constraints which characterize Einsteinian quartic gravities (6.6).

## E Wald formalism: examples

In this appendix we evaluate explicitly the expressions found in section 10 for some relevant theories, namely: Einstein gravity,  $f(R)$  gravity, general quadratic gravities, and Lovelock theories. Note that the expressions below are valid for any background metric  $g_{\mu\nu}$  and vector field  $\xi^\mu$ . Some of these formulas — but not all of them — can also be found in [41, 47, 84–86, 90, 113, 114]. The following identities are frequently used

$$\frac{\partial R_{\mu\alpha\beta\nu}}{\partial R_{\sigma\rho\lambda\eta}} = \frac{1}{2} \left[ \delta_\mu^{[\sigma} \delta_\alpha^{\rho]} \delta_\beta^{[\lambda} \delta_\nu^{\eta]} + \delta_\mu^{[\lambda} \delta_\alpha^{\eta]} \delta_\beta^{[\sigma} \delta_\nu^{\rho]} \right], \quad \frac{\partial R_{\rho\sigma}}{\partial R_{\mu\alpha\beta\nu}} = \delta_{(\rho}^{[\alpha} g^{\mu][\beta} \delta_{\sigma]}^{\nu]}, \tag{E.1}$$

$$\frac{\partial R}{\partial R_{\mu\alpha\beta\nu}} = g^{\beta[\mu} g^{\alpha]\nu}, \quad \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}, \quad \delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \delta g_{\mu\nu}. \tag{E.2}$$

### Einstein gravity

$$\mathbf{L} = \frac{1}{2\kappa} \epsilon (-2\Lambda_0 + R), \tag{E.3}$$

$$P^{\mu\alpha\beta\nu} = \frac{1}{4\kappa} (g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}), \tag{E.4}$$

$$\mathcal{E}_{\mu\nu} = \frac{1}{2\kappa} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_0 g_{\mu\nu} \right), \tag{E.5}$$

$$\Theta = \frac{1}{2\kappa} \epsilon_\mu (g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}) \nabla_\nu \delta g_{\alpha\beta}, \tag{E.6}$$

$$\omega = \epsilon_\mu S^{\mu\alpha\beta\nu\rho\sigma} (\delta_1 g_{\rho\sigma} \nabla_\nu \delta_2 g_{\alpha\beta} - \delta_2 g_{\rho\sigma} \nabla_\nu \delta_1 g_{\alpha\beta}) , \quad (\text{E.7})$$

$$\begin{aligned} S^{\mu\alpha\beta\nu\rho\sigma} = & \frac{1}{2\kappa} \left[ -g^{\mu(\alpha} g^{\beta)(\rho} g^{\sigma)\nu} + \frac{1}{2} g^{\mu(\alpha} g^{\beta)\nu} g^{\rho\sigma} \right. \\ & \left. + \frac{1}{2} g^{\alpha\beta} g^{\mu(\rho} g^{\sigma)\nu} + \frac{1}{2} g^{\mu\nu} g^{\alpha(\rho} g^{\sigma)\beta} - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} g^{\rho\sigma} \right] , \end{aligned} \quad (\text{E.8})$$

$$\mathbf{J}_\xi = \epsilon_\mu \left[ \frac{1}{\kappa} \nabla_\nu (\nabla^{[\nu} \xi^{\mu]}) + 2\mathcal{E}^\mu{}_\nu \xi^\nu \right] , \quad (\text{E.9})$$

$$\mathbf{Q}_\xi = -\frac{1}{2\kappa} \epsilon_{\mu\nu} \nabla^\mu \xi^\nu , \quad (\text{E.10})$$

$$\begin{aligned} \mathbf{k}_\xi = & \frac{1}{2\kappa} \epsilon_{\mu\nu} \left[ (g^{\mu\alpha} \nabla^\beta \xi^\nu - \frac{1}{2} g^{\alpha\beta} \nabla^\mu \xi^\nu) \delta g_{\alpha\beta} \right. \\ & \left. + (g^{\mu\alpha} g^{\nu\lambda} \xi^\beta - g^{\mu\alpha} g^{\beta\lambda} \xi^\nu + g^{\alpha\beta} g^{\mu\lambda} \xi^\nu) \nabla_\lambda \delta g_{\alpha\beta} \right] . \end{aligned} \quad (\text{E.11})$$

$f(R)$  gravity

$$\mathbf{L} = \frac{1}{2\kappa} \epsilon f(R) , \quad (\text{E.12})$$

$$P^{\mu\alpha\beta\nu} = \frac{1}{4\kappa} f'(R) (g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}) , \quad (\text{E.13})$$

$$C^{\sigma\rho\lambda\eta}_{\mu\alpha\beta\nu} = \frac{1}{8\kappa} f''(R) (g_{\mu\beta} g_{\alpha\nu} - g_{\mu\nu} g_{\alpha\beta}) (g^{\sigma\lambda} g^{\rho\eta} - g^{\sigma\eta} g^{\rho\lambda}) , \quad (\text{E.14})$$

$$\mathcal{E}_{\mu\nu} = \frac{1}{2\kappa} \left( f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) f'(R) \right) , \quad (\text{E.15})$$

$$\Theta = f'(R) \Theta_{\text{Ein}} + \frac{1}{2\kappa} \epsilon_\mu (g^{\alpha\beta} \nabla^\mu f'(R) - g^{\beta\mu} \nabla^\alpha f'(R)) \delta g_{\alpha\beta} , \quad (\text{E.16})$$

$$\begin{aligned} \omega = & f'(R) \omega_{\text{Ein}} + \frac{1}{2\kappa} \epsilon_\mu \left[ \frac{1}{2} g^{\mu\beta} g^{\alpha\nu} g^{\rho\sigma} \delta_1 g_{\rho\sigma} \delta_2 g_{\alpha\beta} \nabla_\nu f'(R) \right. \\ & \left. + (g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}) (\delta_1 (f'(R)) \nabla_\nu \delta_2 g_{\alpha\beta} - \delta_1 (\nabla_\nu f'(R)) \delta_2 g_{\alpha\beta}) - [1 \leftrightarrow 2] \right] , \end{aligned} \quad (\text{E.17})$$

$$\mathbf{J}_\xi = \epsilon_\mu \left[ \frac{1}{\kappa} \nabla_\nu (f'(R) \nabla^{[\nu} \xi^{\mu]}) + 2\xi^{[\nu} \nabla^{\mu]} f'(R) \right] + 2\mathcal{E}^\mu{}_\nu \xi^\nu , \quad (\text{E.18})$$

$$\mathbf{Q}_\xi = -\frac{1}{2\kappa} \epsilon_{\mu\nu} [f'(R) \nabla^\mu \xi^\nu + 2\xi^\mu \nabla^\nu f'(R)] , \quad (\text{E.19})$$

$$\begin{aligned} \mathbf{k}_\xi = & f'(R) \mathbf{k}_{\xi, \text{Ein}} - \frac{1}{2\kappa} \epsilon_{\mu\nu} \left[ \nabla^\mu \xi^\nu \delta f'(R) \right. \\ & \left. - 2g^{\mu\alpha} \xi^\nu \delta (\nabla_\alpha f'(R)) + g^{\mu\alpha} \xi^\nu \nabla^\beta (f'(R)) \delta g_{\alpha\beta} \right] . \end{aligned} \quad (\text{E.20})$$

## Quadratic gravity

$$\mathbf{L} = \epsilon \left\{ \frac{1}{2\kappa} (-2\Lambda_0 + R) + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right\}. \quad (\text{E.21})$$

Recall that Gauss-Bonnet gravity can be obtained by setting<sup>46</sup>  $\alpha_1 = \alpha_3 = -\frac{1}{4}\alpha_2 = \alpha$ . That theory has the interesting feature — shared by all Lovelock gravities — that  $P^{\mu\alpha\beta\nu}$  is divergence-free in all indices, *e.g.*,  $\nabla_\mu P^{\mu\alpha\beta\nu} = 0$ . Hence, all derivatives of curvature tensors should cancel in that case for the forms below, which provides a simple check for our expressions.

The first derivative of the Lagrangian with respect to the Riemann tensor as defined in (2.2) is

$$\begin{aligned} P^{\mu\alpha\beta\nu} &= \left( \frac{1}{4\kappa} + \alpha_1 R \right) (g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}) \\ &\quad + \frac{1}{2} \alpha_2 (R^{\mu\beta} g^{\alpha\nu} - R^{\alpha\beta} g^{\mu\nu} - R^{\mu\nu} g^{\alpha\beta} + R^{\alpha\nu} g^{\mu\beta}) + 2\alpha_3 R^{\mu\alpha\beta\nu}, \end{aligned} \quad (\text{E.22})$$

and its divergence reads

$$\nabla_\mu P^{\mu\alpha\beta\nu} = \left( 2\alpha_1 + \frac{1}{2}\alpha_2 \right) g^{\alpha[\nu} \nabla^{\beta]} R + (\alpha_2 + 4\alpha_3) \nabla^{[\beta} R^{\nu]\alpha}, \quad (\text{E.23})$$

where we have used the following identities:  $\nabla^\nu R_{\mu\nu} = \frac{1}{2}\nabla_\mu R$  and  $\nabla^\rho R_{\mu\nu\sigma\rho} = -2\nabla_{[\mu} R_{\nu]\sigma}$ . These can be derived from the second Bianchi identity, and will be frequently employed to simplify our expressions below. Notice that the divergence indeed vanishes for Gauss-Bonnet gravity.

From this we find for the tensor defined in (2.4)

$$\begin{aligned} C_{\mu\alpha\beta\nu}^{\sigma\rho\lambda\eta} &= \frac{1}{2} \alpha_1 (g_{\mu\beta} g_{\alpha\nu} - g_{\mu\nu} g_{\alpha\beta}) (g^{\sigma\lambda} g^{\rho\eta} - g^{\sigma\eta} g^{\rho\lambda}) + 2\alpha_2 \delta_{(\tau}^{[\sigma} g^{\rho][\lambda} \delta_{\epsilon)}^{\eta]} \delta_{[\mu}^\tau g_{\alpha][\beta} \delta_{\nu]}^\epsilon \\ &\quad + \alpha_3 \left( \delta_\mu^{[\sigma} \delta_\alpha^{\rho]} \delta_\beta^{[\lambda} \delta_\nu^{\eta]} + \delta_\mu^{[\lambda} \delta_\alpha^{\eta]} \delta_\beta^{[\sigma} \delta_\nu^{\rho]} \right). \end{aligned} \quad (\text{E.24})$$

The equations of motion for quadratic gravity read

$$\begin{aligned} \mathcal{E}_{\mu\nu} &= \frac{1}{2\kappa} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda_0 g_{\mu\nu} \right) + \alpha_1 \left( 2R R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^2 - 2\nabla_\mu \nabla_\nu R + 2g_{\mu\nu} \square R \right) \\ &\quad + \alpha_2 \left( R_{\mu\rho} R_\nu{}^\rho + R_{\rho\sigma} R_\mu{}^\rho{}_\nu{}^\sigma + \frac{1}{2} g_{\mu\nu} (\square R - R_{\rho\sigma} R^{\rho\sigma}) - \nabla_{(\mu} \nabla_{\nu)} R + \square R_{\mu\nu} \right) \\ &\quad + \alpha_3 \left( 2R_{\mu\rho\sigma\lambda} R_\nu{}^{\rho\sigma\lambda} - \frac{1}{2} g_{\mu\nu} R_{\rho\sigma\alpha\beta} R^{\rho\sigma\alpha\beta} - 4\nabla^\rho \nabla^\sigma R_{\mu\rho\sigma\nu} \right). \end{aligned} \quad (\text{E.25})$$

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<sup>46</sup>Note that in this section the couplings are not assumed to be dimensionless. This avoids some clutter in the already messy expressions

The symplectic potential form (10.4) is

$$\begin{aligned}\Theta = \epsilon_\mu \Big[ & \left( \frac{1}{2\kappa} + 2\alpha_1 R \right) (g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}) \nabla_\nu \delta g_{\alpha\beta} + 4\alpha_1 (g^{\beta[\alpha} \nabla^{\mu]} R) \delta g_{\alpha\beta} \\ & + 2\alpha_2 \left( R^{\beta[\mu} \nabla^{\alpha]} \delta g_{\alpha\beta} + g^{\beta[\mu} R^{\alpha]\nu} \nabla_\nu \delta g_{\alpha\beta} + \nabla^{[\mu} R^{\alpha]\beta} \delta g_{\alpha\beta} - \frac{1}{2} g^{\beta[\mu} \nabla^{\alpha]} R \delta g_{\alpha\beta} \right) \\ & + 4\alpha_3 (R^{\mu\alpha\beta\nu} \nabla_\nu \delta g_{\alpha\beta} + 2\nabla^{[\mu} R^{\alpha]\beta} \delta g_{\alpha\beta}) \Big].\end{aligned}\quad (\text{E.26})$$

For Gauss-Bonnet gravity, this reduces to

$$\begin{aligned}\Theta_{\text{GB}} = \epsilon_\mu \Big[ & \left( \frac{1}{2\kappa} + 2\alpha R \right) (g^{\mu\beta} g^{\alpha\nu} - g^{\mu\nu} g^{\alpha\beta}) \nabla_\nu \delta g_{\alpha\beta} \\ & - 8\alpha \left( R^{\beta[\mu} \nabla^{\alpha]} \delta g_{\alpha\beta} + g^{\beta[\mu} R^{\alpha]\nu} \nabla_\nu \delta g_{\alpha\beta} \right) + 4\alpha R^{\mu\alpha\beta\nu} \nabla_\nu \delta g_{\alpha\beta} \Big].\end{aligned}\quad (\text{E.27})$$

Note that this object was previously computed in equation (70) of [41]. We observe that our expression above differs from their result by a total derivative

$$\Theta_{\text{IW}} - \Theta_{\text{GB}} = 8\alpha \epsilon_\mu \nabla_\nu (R^{\alpha[\mu} g^{\nu]\beta} \delta g_{\alpha\beta}), \quad (\text{E.28})$$

but only if the sign of the second to last term in their formula (70) is modified — to be explicit: this term should be “ $+4(\nabla^e R^{df})\delta g_{ef}$ ”. Hence, we suspect there is a typo in their expression. This is consistent with [115], where the symplectic potential was also computed for quadratic gravity. Restricting their formula (3.7) for the symplectic potential to Gauss-Bonnet indeed produces the Iyer-Wald symplectic potential *with* the corrected sign.

The symplectic current form (10.7) reads

$$\begin{aligned}\omega = \epsilon_\mu \Big[ & (1 + 4\kappa\alpha_1 R) S_{\text{Ein}}^{\mu\alpha\beta\nu\rho\sigma} \delta_1 g_{\rho\sigma} \nabla_\nu \delta_2 g_{\alpha\beta} \\ & + \alpha_1 (g^{\mu\beta} g^{\alpha\nu} g^{\rho\sigma} \delta_1 g_{\rho\sigma} \delta_2 g_{\alpha\beta} \nabla_\nu R + 4g^{\mu[\beta} g^{\nu]\alpha} (\delta_1(R) \nabla_\nu \delta_2 g_{\alpha\beta} - \delta_1(\nabla_\nu R) \delta_2 g_{\alpha\beta})) \\ & + \alpha_2 \left( 2\delta_1(R^{\beta[\mu} g^{\alpha]\nu}) - 2\delta_1(R^{\nu[\mu} g^{\alpha]\beta}) + (R^{\rho[\mu} g^{\nu]\beta} g^{\alpha\sigma} - R^{\beta(\mu} g^{\nu)\rho} g^{\alpha\sigma} + R^{\rho[\mu} g^{\alpha]\nu} g^{\beta\sigma} \right. \\ & + R^{\nu(\mu} g^{\alpha)\rho} g^{\beta\sigma} + R^{\alpha[\beta} g^{\mu]\rho} g^{\nu\sigma} + R^{\rho[\alpha} g^{\mu]\beta} g^{\nu\sigma} + R^{\beta[\mu} g^{\alpha]\nu} g^{\rho\sigma} + R^{\nu[\alpha} g^{\mu]\beta} g^{\rho\sigma}) \delta_1 g_{\rho\sigma} \Big) \nabla_\nu \delta_2 g_{\alpha\beta} \\ & - \alpha_2 \left( 2\delta_1(\nabla^{[\alpha} R^{\mu]\beta}) + \frac{1}{2} g^{\beta[\mu} \nabla^{\alpha]} R \right) + (\nabla^{[\alpha} R^{\mu]\beta} + \frac{1}{2} g^{\beta[\mu} \nabla^{\alpha]} R) g^{\rho\sigma} \delta_1 g_{\rho\sigma} \Big) \delta_2 g_{\alpha\beta} \\ & + 4\alpha_3 \left( \delta_1 R^{\mu\alpha\beta\nu} + (R^{\mu\alpha\beta(\nu} g^{\rho)\sigma} - R^{\mu(\alpha\nu)\rho} g^{\beta\sigma}) \delta_1 g_{\rho\sigma} \right) \nabla_\nu \delta_2 g_{\alpha\beta} \\ & + 4\alpha_3 (2\delta_1 \nabla^{[\mu} R^{\alpha]\beta} + g^{\rho\sigma} \delta_1 g_{\rho\sigma} \nabla^{[\mu} R^{\alpha]\beta}) \delta_2 g_{\alpha\beta} \Big] - [1 \leftrightarrow 2].\end{aligned}\quad (\text{E.29})$$

One can check that all terms involving derivatives acting on curvature tensors cancel for Gauss-Bonnet gravity.

The Noether current (10.23) and charge (10.22) are given by

$$\mathbf{J}_\xi = \epsilon_\mu \left[ \nabla_\nu \left[ \left( \frac{1}{\kappa} + 4\alpha_1 R \right) \nabla^{[\nu} \xi^{\mu]} + 8\alpha_1 \xi^{[\nu} \nabla^{\mu]} R + 4\alpha_2 \left( R^{\rho[\nu} \nabla_{\rho} \xi^{\mu]} + 2\xi^{[\nu} \nabla_{\rho} R^{\mu]\rho} \right) \right. \right. \\ \left. \left. - 4\alpha_3 \left( R^{\mu\nu\rho\sigma} \nabla_{\rho} \xi_{\sigma} + 2\xi_{\rho} \nabla_{\sigma} R^{\mu\nu\rho\sigma} \right) \right] + 2\mathcal{E}^{\mu}{}_{\nu} \xi^{\nu} \right], \quad (\text{E.30})$$

$$\mathbf{Q}_\xi = -\epsilon_{\mu\nu} \left[ \left( \frac{1}{2\kappa} + 2\alpha_1 R \right) \nabla^{\mu} \xi^{\nu} + 4\alpha_1 \xi^{\mu} \nabla^{\nu} R \right. \\ \left. + 2\alpha_2 \left( R^{\mu}{}_{\rho} \nabla^{[\rho} \xi^{\nu]} + 2\xi^{[\mu} \nabla^{\rho]} R^{\nu}{}_{\rho} \right) + 2\alpha_3 \left( R^{\mu\nu\rho\sigma} \nabla_{\rho} \xi_{\sigma} + 2\xi_{\rho} \nabla_{\sigma} R^{\mu\nu\rho\sigma} \right) \right]. \quad (\text{E.31})$$

For Gauss-Bonnet gravity, we find the same expression as in [41], namely

$$\mathbf{Q}_{\xi, \text{GB}} = -\epsilon_{\mu\nu} \left[ \left( \frac{1}{2\kappa} + 2\alpha R \right) \nabla^{\mu} \xi^{\nu} - 8\alpha R^{\mu}{}_{\rho} \nabla^{[\rho} \xi^{\nu]} + 2\alpha R^{\mu\nu\rho\sigma} \nabla_{\rho} \xi_{\sigma} \right]. \quad (\text{E.32})$$

Finally, the Iyer-Wald surface charge (10.29) is

$$\mathbf{k}_\xi = (1 + 4\kappa\alpha_1) \mathbf{k}_{\xi, \text{Ein}} \\ + \epsilon_{\mu\nu} \left[ 2\alpha_1 \left( -\nabla^{\mu} \xi^{\nu} \delta R + 2\xi^{\nu} \delta(\nabla^{\mu} R) + g^{\mu\alpha} \xi^{\nu} \nabla^{\beta} R \delta g_{\alpha\beta} \right) \right. \\ + \alpha_2 \left( \nabla^{[\nu} \xi^{\lambda]} \delta R^{\mu}{}_{\lambda} + g^{\mu\lambda} g_{\alpha\beta} \nabla^{[\nu} \xi^{\alpha]} \delta R^{\beta}{}_{\lambda} + 2\xi^{\lambda} \delta(\nabla^{\mu} R^{\nu}{}_{\lambda}) + \xi^{\nu} \delta(\nabla^{\mu} R) \right) \\ + \alpha_2 \left( 2R^{\mu[\alpha} g^{\lambda]\nu} \nabla^{\beta} \xi_{\lambda} - g^{\alpha\beta} R^{\mu[\rho} g^{\sigma]\nu} \nabla_{\rho} \xi_{\sigma} + 2\xi^{\nu} \nabla^{[\alpha} R^{\mu]\beta} + \xi^{\nu} \nabla^{[\alpha} R g^{\mu]\beta} \right. \\ \left. - 2\xi_{\rho} g^{\alpha\beta} \nabla_{\sigma} R^{\mu[\rho} g^{\sigma]\nu} \right) \delta g_{\alpha\beta} - 2\alpha_2 \left( 2\xi^{\nu} R^{\alpha(\mu} g^{\lambda)\beta} + \xi^{\alpha} R^{\nu[\beta} g^{\lambda]\mu} \right) \nabla_{\lambda} \delta g_{\alpha\beta} \\ - \alpha_3 \left( \nabla^{\alpha} \xi^{\beta} \delta R^{\mu\nu}{}_{\alpha\beta} + g^{\mu\alpha} g^{\nu\beta} \nabla_{\rho} \xi_{\sigma} \delta R^{\rho\sigma}{}_{\alpha\beta} - 8\xi^{\lambda} \delta(\nabla^{\mu} R^{\nu}{}_{\lambda}) \right) \\ + 2\alpha_3 \left( R^{\mu\nu\alpha\lambda} \nabla^{\beta} \xi_{\lambda} - \frac{1}{2} R^{\mu\nu\rho\sigma} g^{\alpha\beta} \nabla_{\rho} \xi_{\sigma} - 4\xi^{\nu} \nabla^{[\mu} R^{\alpha]\beta} + 2\xi_{\lambda} \nabla^{\mu} R^{\nu\lambda} g^{\alpha\beta} \right) \delta g_{\alpha\beta} \\ \left. - 2\alpha_3 \left( \xi^{\alpha} R^{\mu\nu\lambda\beta} + 2\xi^{\nu} R^{\mu\alpha\beta\lambda} \right) \nabla_{\lambda} \delta g_{\alpha\beta} \right]. \quad (\text{E.33})$$

Again, it is straightforward to verify that all terms involving derivatives of curvature tensors cancel for Gauss-Bonnet gravity.

### Lovelock gravity

The Lagrangian of Lovelock gravity is

$$\mathbf{L} = \frac{1}{2\kappa} \epsilon \sum_{k=0}^{[D/2]} c_k \mathcal{L}_k \quad \text{with} \quad \mathcal{L}_k = \frac{1}{2^k} \delta^{\mu_1 \nu_1 \dots \mu_k \nu_k}_{\alpha_1 \beta_1 \dots \alpha_k \beta_k} R_{\mu_1 \nu_1}{}^{\alpha_1 \beta_1} \dots R_{\mu_k \nu_k}{}^{\alpha_k \beta_k}, \quad (\text{E.34})$$

where the  $c_k$  are arbitrary constants. The objects defined in (2.2) and (2.4) read respectively

$$P^{\mu\alpha\beta\nu} = \frac{1}{2\kappa} \sum_{k=0}^{\lfloor D/2 \rfloor} c_k P_{(k)}^{\mu\alpha\beta\nu}, \quad P_{(k)\alpha\beta}^{\mu\nu} = \frac{k}{2^k} \delta_{\alpha\beta\alpha_2\beta_2\ldots\alpha_k\beta_k}^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k} R_{\mu_2\nu_2}^{\alpha_2\beta_2} \ldots R_{\mu_k\nu_k}^{\alpha_k\beta_k}, \quad (\text{E.35})$$

$$C_{\mu\alpha\beta\nu}^{\sigma\rho\lambda\eta} = \frac{1}{2\kappa} \sum_{k=0}^{\lfloor D/2 \rfloor} \frac{k(k-1)c_k}{2^k} g_{\beta\gamma} g_{\nu\delta} g^{\lambda\chi} g^{\eta\xi} \delta_{\mu\alpha\chi\xi\alpha_3\beta_3\ldots\alpha_k\beta_k}^{\sigma\rho\gamma\delta\mu_3\nu_3\ldots\mu_k\nu_k} R_{\mu_3\nu_3}^{\alpha_3\beta_3} \ldots R_{\mu_k\nu_k}^{\alpha_k\beta_k}.$$

The equations of motion are

$$\mathcal{E}_{\mu\nu} = \frac{1}{2\kappa} \sum_{k=0}^{\lfloor D/2 \rfloor} c_k \mathcal{E}_{\mu\nu}^{(k)} \quad \text{with} \quad \mathcal{E}^{(k)\mu}{}_{\nu} = \frac{-1}{2^{k+1}} \delta_{\nu\alpha_1\beta_1\ldots\alpha_k\beta_k}^{\mu\rho_1\sigma_1\ldots\rho_k\sigma_k} R_{\rho_1\sigma_1}^{\alpha_1\beta_1} \ldots R_{\rho_k\sigma_k}^{\alpha_k\beta_k}. \quad (\text{E.36})$$

Both tensors (E.35) and (E.36) are divergence-free in all indices, *e.g.*,  $\nabla_\mu P^{\mu\alpha\beta\nu} = 0$ ,  $\nabla^\mu \mathcal{E}_{\mu\nu} = 0$ . Note that the equations of motion are second order in the metric, as is well known for Lovelock gravity.

The rest of the relevant quantities read

$$\Theta = \frac{1}{2\kappa} \epsilon_\mu \sum_{k=0}^{\lfloor D/2 \rfloor} \frac{k c_k}{2^{k-1}} \delta_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_k\beta_k}^{\mu\nu_1\mu_2\nu_2\ldots\mu_k\nu_k} R_{\mu_2\nu_2}^{\alpha_2\beta_2} \ldots R_{\mu_k\nu_k}^{\alpha_k\beta_k} g^{\alpha_1\lambda} \nabla^{\beta_1} \delta g_{\nu_1\lambda}, \quad (\text{E.37})$$

$$\begin{aligned} \omega &= \frac{1}{2\kappa} \epsilon_\mu \sum_{k=0}^{\lfloor D/2 \rfloor} \frac{k c_k}{2^k} \left[ \delta_{\gamma\delta\alpha_2\beta_2\ldots\alpha_k\beta_k}^{\mu\alpha\mu_2\nu_2\ldots\mu_k\nu_k} \left( 2(k-1) g^{\beta\gamma} g^{\nu\delta} \delta_1 R_{\mu_2\nu_2}^{\alpha_2\beta_2} \right. \right. \\ &\quad \left. \left. + (g^{\beta\gamma} g^{\rho\sigma} g^{\nu\delta} + g^{\beta\delta} g^{\rho\gamma} g^{\nu\sigma} - g^{\beta\sigma} g^{\rho\gamma} g^{\nu\delta}) \delta_1 g_{\rho\sigma} R_{\mu_2\nu_2}^{\alpha_2\beta_2} \right) \right. \\ &\quad \left. + g^{\beta\sigma} g^{\rho\gamma} g^{\alpha\delta} \delta_1 g_{\rho\sigma} \delta_{\gamma\delta\alpha_2\beta_2\ldots\alpha_k\beta_k}^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k} R_{\mu_2\nu_2}^{\alpha_2\beta_2} \right] R_{\mu_3\nu_3}^{\alpha_3\beta_3} \ldots R_{\mu_k\nu_k}^{\alpha_k\beta_k} \nabla_\nu \delta_2 g_{\alpha\beta} - [1 \leftrightarrow 2], \end{aligned} \quad (\text{E.38})$$

$$\begin{aligned} \mathbf{J}_\xi &= \frac{1}{2\kappa} \epsilon_\mu \sum_{k=0}^{\lfloor D/2 \rfloor} c_k \left[ \nabla_\nu \left( \frac{-k}{2^{k-1}} \delta_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_k\beta_k}^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k} R_{\mu_2\nu_2}^{\alpha_2\beta_2} \ldots R_{\mu_k\nu_k}^{\alpha_k\beta_k} \nabla^{\alpha_1} \xi^{\beta_1} \right) \right. \\ &\quad \left. + 2 \mathcal{E}^{(k)\mu}{}_{\nu} \xi^\nu \right], \end{aligned} \quad (\text{E.39})$$

$$\mathbf{Q}_\xi = -\frac{1}{2\kappa} \epsilon_{\mu\nu} \sum_{k=0}^{\lfloor D/2 \rfloor} \frac{k c_k}{2^k} \delta_{\alpha_1\beta_1\alpha_2\beta_2\ldots\alpha_k\beta_k}^{\mu\nu\mu_2\nu_2\ldots\mu_k\nu_k} R_{\mu_2\nu_2}^{\alpha_2\beta_2} \ldots R_{\mu_k\nu_k}^{\alpha_k\beta_k} \nabla^{\alpha_1} \xi^{\beta_1}, \quad (\text{E.40})$$



$$\begin{aligned}
\mathbf{k}_\xi = & \frac{1}{2\kappa} \epsilon_{\mu\nu} \sum_{k=0}^{\lfloor D/2 \rfloor} \frac{k C_k}{2^k} \left[ \delta_{\gamma\delta\alpha_2\beta_2\dots\alpha_k\beta_k}^{\mu\nu\mu_2\nu_2\dots\mu_k\nu_k} \left( - (k-1) \nabla^\gamma \xi^\delta \delta R_{\mu_2\nu_2}{}^{\alpha_2\beta_2} \right. \right. \\
& + \left( g^{\rho\gamma} \nabla^\sigma \xi^\delta \delta g_{\rho\sigma} - \frac{1}{2} g^{\rho\sigma} \nabla^\gamma \xi^\delta \delta g_{\rho\sigma} - \xi^\rho g^{\sigma\delta} \nabla^\gamma \delta g_{\rho\sigma} \right) R_{\mu_2\nu_2}{}^{\alpha_2\beta_2} \Big) \\
& \left. - 2\xi^\alpha g^{\gamma\sigma} \nabla^\delta \delta g_{\rho\sigma} \delta_{\gamma\delta\alpha_2\beta_2\dots\alpha_k\beta_k}^{\mu\rho\mu_2\nu_2\dots\mu_k\nu_k} R_{\mu_2\nu_2}{}^{\alpha_2\beta_2} \right] R_{\mu_3\nu_3}{}^{\alpha_3\beta_3} \dots R_{\mu_k\nu_k}{}^{\alpha_k\beta_k} .
\end{aligned} \tag{E.41}$$

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